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LIST OF SYMBOLS

Notation	Meaning

MFNDE	Fractional multi-order nonlinear differential equation
${}_{a}I_{x}^{\alpha}$	The left Riemann-Liounville fractional integral .
${}_{b}I_{x}^{\alpha}$	The right Riemann-Liounville fractional integral .
${}_{a}D_{x}^{\alpha}$	The left Riemann-Liounville fractional derivative .
${}_{b}D_{x}^{\alpha}$	The right Riemann-Liounville fractional derivative .
${}_{a}^{c}D_{x}^{\alpha}$	The left Caputo fractional integral .
${}_{x}^{c}D_{b}^{\alpha}$	The right Caputo fractional integral .
$T_n(x)$	Chebyshev polynomial of first kind .
$V_n(x)$	Chebyshev polynomial of third kind .
$W_n(x)$	Chebyshev polynomial of fourth kind .
$T_n^*(x)$	Shifted Chebyshev polynomial of first kind .
$T_i^p(x)$	Shifted pseudo-spectral Chebyshev polynomial of first kind .
$V_n^*(x)$	Shifted Chebyshev polynomial of third kind .
$W_n^*(x)$	Shifted Chebyshev polynomial of fourth kind .
$\omega_i(x)$	Weight function for Chebyshev polynomial, i=1,2,3,4 .
$\omega_i^*(x)$	Weight function for shifted Chebyshev polynomial, i=1,2,3,4
$\omega_1^p(x)$	Weight function for shifted pseudo-spectral Chebyshev polynomial of first kind .
Δ^{α}	Operational matrix of fractional derivative of order α>0 .
$\Psi_{a,b}(t)$	Wavelet functions .
$\Psi_{nm}^{i}(t)$	Shifted Chebyshev wavelets , i=1,2,3,4 .
$\omega_{i,n}^*(2^k t - n)$	Weight function for shifted Chebyshev wavelets , i=1,2,3,4 .
$\bar{T}_n^{\alpha}(x)$	Shifted Chebyshev polynomial first kind of order α .
$\bar{V}_n^{\alpha}(x)$	Shifted Chebyshev polynomial third kind of order α .
$\bar{W}_n^{\alpha}(x)$	Shifted Chebyshev polynomial fourth kind of order α .
$\bar{\Delta}^{\lambda}$	Fractional matrix of order different than order of the equation.

$\bar{\Delta}^{\alpha}$	Fractional matrix of order equal the order of equation.

ABSTRACT

This thesis, constancies the numerical solutions of multi-fractional order nonlinear differential problems with initial (mixed boundary) conditions which have been considered in details. The fractional operational matrices of fractional derivative $\beta > 0$ have been studied and developed on types of shifted chebyshev polynomials and shifted chebyshev wavelets as well as fractional order chebyshev polynomials with order $\alpha > 0$ and presented the relation between these types, they are given as an operational matrices of fractional derivatives as well as, we have given two types of fractional operational matrices for fractional derivatives depended on $\beta > 0$values. Also the coupled fractional orders such that one of them is originally and others are axillary order such that one a fractional operational matrix to find the best approximate solution of multi-fractional order nonlinear differential problems with mixed boundary value. The rule of the order of fractional polynomial is presented as important parameter for given exact solution and is illustrated in some examples.

Contents

INTRODUCTION

Many phenomena in engineering physics, chemistry, and other sciences can be described successfully by models that use mathematical tools of fractional calculus, i.e. the theory of derivatives and integrals of non-integer order [27,45,46]. For example, they have been successfully used in modeling frequency dependent damping behavior of many viscoelastic materials. There are numerous research which demonstrate the applications of fractional derivatives in the areas of electrochemical processes, dielectric polarization, colored noise, and chaos. [19]

The numerical solution of differential equations of integer order has been a hot topic in numerical and computational mathematics for a long time. The solution of fractional differential equations has been recently studied by numerous authors. However, the state of the art is fearless advanced for general fractional order differential equations. Moreover, to the best of the authors knowledge, very few algorithms for the numerical solution of multi-order fractional differential equations have been suggested [10,35,47], particularly algorithms for analytical solutions and approximate solutions of nonlinear multi-order fractional differential equations. [19]

Fractional-order differential equations, as generalizations of classical integer-order differential equations, are increasingly used to model some problems in fluid, mechanics, viscoelasticity, biology, physics, engineering, and other applications. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [6,13,30,37,38,39]. The solutions of fractional order differential equations are much involved, because in general, there exists no method that yields an exact solution for fractional order differential equations, and only approximate solutions can be derived using linearization or perturbation methods. Several methods have been suggested to solve fractional differential equations [see 26]. [16]

For multi-order fractional differential equation, an operational matrix of fractional integration in complex way is studied in [19],[20],[44]. Adams method is used to solve multi-order fractional differential equations and their numerical in [8], Euler's method, product trapeziodal quadrature formula, product considered in [47], V.Gejji and H.Jafari are studied Adomian decomposition method in(2007), the variational iteration method in [48],[50], the generalized Laguerre polynomials is studied in [21], Legendre pseudo-spectral method in [19], and Chebyshev wavelets with integration fractional operational matrices are considered in [26].

The Chebyshev polynomials are one of the most useful polynomials which are suitable in numerical analysis including polynomial approximation, integral and differential equations and spectral methods for partial differential equations [4,5,12,34] .(see[33])

In recent years, wavelets have received considerable attention by researchers in different fields of science and engineering. One advantage of wavelet analysis is the ability to perform local analysis [31]. Wavelet analysis is able to reveal signal aspects that other analysis methods miss, such as trends, breakdown point and discontinuities. In comparison with other orthogonal function, multiresolution analysis aspect of wavelets permits the accurate representation of a variety of functions and operators. In other words, we can change M and K simultaneously to get more accurate solution. Another benefit of wavelet method for solving equations is that after discreting the coefficients matrix of algebraic equations is sparse. So the use of wavelet methods for solving equations is computationally efficient. In addition, the solution is convergent. The operational matrix of fractional order integration for Chebyshev wavelet, Legendre wavelet, and haar wavelet has been introduced in [15,26,31] to solve the differential equations of fractional order, [18].

However, few papers have reported applications of wavelets in solving fractional differential equations [19,25,28,41,49]. In view of successful application of wavelet operational matrices in numerical solution of integral and differential equations [22,40,42], together with the characteristics of wavelet functions, we believe that they should be applicable in solving multi-order fractional differential equations [16].

One of the attractive concepts in the initial and boundary value problems is differentiation and integration of fractional order (K.Diethelm),(Fox),(K.B.Oldham). Many researchers extend classical methods in studies of differential and integral equations of integer order to fractional type of these problems (X.Li),(A.Saaadatmandi and others). One of the wide classes of researches focuses on constructing the operational matrix of derivative in some spectral methods. Recently, a lot of attention has been devoted to construct operational matrix of fractional derivative [4,23,43].[33]

In this work , we will set a new modified operational matrices of shifted Chebyshev polynomial and Chebshev wavelets, some of them is new formula appeared in [29], to solving multi-fractional order of nonlinear differential equations, also we set a new modified fractional operational matrices for using fractional Chebyshev polynomials of shifted (first, third, fourth) kinds for fractional order of polynomial different from or equal to the fractional order of multi-fractional order nonlinear differential equations with mixed boundary conditions.

The new relations between the Chebyshev wavelets kinds also are given to complete all the useful of operational matrices.

The obtained solutions of above methods, show that the operational matrices and fractional operational matrices are very convenient and efficient and only few calculations to give high accurate and may lead to exact solutions.

This thesis consists of three chapters.

In chapter one, we study some of special functions including gamma and beta function. The definitions are related to fractional concepts. In this chapter, the kinds of Chebyshev polynomials with their relations and some kinds of chebyshev wavelets, their function approximation and some illustrative examples.

In chapter two, we presented the definitions related to shifted Chebyshev wavelets kinds and give the basic theorems of operational matrices of Chebyshev polynomials kinds and their related in fractional derivative concepts. Also we illustrative some examples for using operational matrices of new formulations for solving multi-order fractional nonlinear differential equations.

Finally, chapter three presents the solution of multi-order fractional nonlinear differential equations with mixed boundary conditions using fractional operational matrices with equal or different type of order of the equation and given the examples to illustrative the methods. Also different fractional operational matrices, one as originally and second as axillary orders are explained with some examples.

The value of x observed in details to explain the activity of approximation of solution with mixed boundary conditions.

It is important to notice that, the calculations are written by using the mathematical software MATHCAD 14.0 .

PRELIMINARIES

1.1 Introduction:

In this chapter, we introduce some definitions of some special known functions and fractional order derivative and their properties. Also the constrictions of Chebyshev polynomials formulation have been given and studied the shifted pseudo-spectral Chabyshev polynomial of first kind with interval[0,L] .

The constriction wavelets chebyshev formulations also have been given with general details of orthogonality which supported by some alterative.

This chapter consists of four sections. In section (1.2), some of special functions are introduced, as well as, some of their important properties. In section (1.3), some types of fractional (derivative and integral) and some important properties. In section (1.4), Chebyshev polynomials of fourth kinds. Finally , in section (1.5), wavelet functions have been presented. Illustrative examples have been given to support all the concepts.

1.2 Some of Special functions:

The basic theory of the special functions which are used later on are given and explained in details.

1.2.1 Gamma Functions,[38]:

The basic functions of the fractional calculus is Euler's gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt \tag{1.1}$$

which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

One of the basic properties of the gamma function is that it satisfies the following functional equation:

1. The gamma function is continuous for all real positive.[24]

2. $\Gamma(z+1) = z\Gamma(z)$. [38]

3. $\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^2}{z(z+1)\cdots(z+n)}$, where $Re(z) > 0, z \neq 0, -1, \ldots$,[38].

4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. [24]

5. $\Gamma(\frac{1}{2}+n) = \frac{(2n)!.\sqrt{\pi}}{2^{2n}.n!}$ $\quad$ n ∈ N . [24]

6. $\Gamma(\frac{1}{2}-n) = \frac{(-4)^n.n!.\sqrt{\pi}}{(2n)!}$ $\quad n \in N$. [37]

7. $\Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!$ $\quad$ for $n = 0,1,2,\cdots$,[38].

8. $\Gamma(-\mathrm{n}) = \frac{-\pi.\csc(\pi x)}{\Gamma(n+1)}$. [16]

9. $\Gamma(xn) = \sqrt{\frac{2\pi}{x}}[\frac{x^n}{\sqrt{2\pi}}]\prod_{k=0}^{x-1}\Gamma(n+\frac{k}{x})$. [37]

Example (1.2.1),[24]:

To evaluate the following function by using gamma function we need to do the following transformation

$f(x) = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Let $x^2 = t$ thus $x = t^{\frac{1}{2}}$ yeilds $dx = \frac{1}{2}t^{\frac{-1}{2}}dt$

$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{\frac{-1}{2}}dt$

$also$, Let $z-1 = \frac{-1}{2}$, then $z = \frac{1}{2}$, we have that,

$$f(x) = \int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{\frac{-1}{2}}dt = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

1.2.2 Beta Function,[38]:

In many cases it is more convenient to use the so-called beta function instead of certain combination of values of the gamma function.

The beta function is usually defined by:-

$$\beta(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}d\tau \quad , \quad Re(z) > 0\,, Re(w) > 0 \qquad (1.2)$$

Let us consider the following integral

$$h_{z,w}(t) = \int_0^t \tau^{z-1}(1-\tau)^{w-1}d\tau.$$

We have that $h_{z,w}(1) = \beta(z,w)$, the Laplace transform is

$$H_{z,w}(S) = \frac{\Gamma(z)}{S^z} \cdot \frac{\Gamma(w)}{S^w} = \frac{\Gamma(z)\cdot\Gamma(w)}{S^{z+w}} \qquad (1.3)$$

The inverse Laplace transform of the right-hand side of (1.3) is,

$$h_{z,w}(t) = \frac{\Gamma(z)\cdot\Gamma(w)}{\Gamma(z+w)} t^{z+w-1}$$

If $t = 1$ we obtain,

$$B(z,w) = \frac{\Gamma(z)\cdot\Gamma(w)}{\Gamma(z+w)} \qquad (1.4)$$

also, we have $B(z,w) = B(w,z)$.

The definition of beta function is valid only for $Re(z) > 0$, $Re(w) > 0$ The formula in (1.4) that obtain the analytical continuation of the beta function for the entire complex plane, if we have the analytically continued gamma function. with the help of the beta function we can establish the following important relationships for the gamma function:

i. $\Gamma(z)\cdot\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$=$\beta(z,1-z)$, Re(z)> 0.

ii. $\Gamma(z)\cdot\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{2z-1}\Gamma(2z) \quad ,for\ \ 2z \neq 0,-1,-2,\cdots$

If z $= n + \frac{1}{2}$, we get $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\cdot\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}\cdot n!}$.

iii. follows from (1.2),If we make the change the variable $\tau = \frac{u}{1+u}$ and $z, w \in Z^+$, we obtain,

$$\beta(x,y) = \int_0^\infty u^{x-1}\,(1+u)^{-(x+y)} du.$$

iv. from [39] we have, $B(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin t)^{2x-1} (\cos t)^{2y-1}\, dt.$

<u>Example(1.2.2),[24]:</u>

To evaluate $f(t) = \int_0^{\frac{\pi}{2}} (\sin(2x))^{2t-1}\, dx$ by using beta function from(iv), we have that f(t)=$\int_0^{\frac{\pi}{2}} (\sin(2x))^{2t-1} \cdot (\cos(2x))^0\, dx$, now ,let

$2m-1=2t-1$, then 2m=2t , we get m=t

$2n-1=0$, then 2n=1 , we get n=$\frac{1}{2}$

Then, $\int_0^{\frac{\pi}{2}}(\sin(2x))^{2t-1}\cdot(\cos(2x))^0\,dx=\frac{1}{2}B\left(t,\frac{1}{2}\right)=\frac{\sqrt{\pi}\cdot\Gamma(t)}{2\Gamma(t+\frac{1}{2})}$

1.3 Some Types of Fractional (Derivative and Integral):

The several definitions of fractional derivatives and fractional integrals, such as Riemann-Liouville, Caputo, Riesz, Riesz Caputo, Wely, Grunwald-letnikov, and Hadaman Chen, etc, are given however, we will present the definitions of the first two of them.

Let $f:[a,b]\rightarrow R$ be a function, α a positive real number, n the integer satisfying $-1\leq\alpha<n$, and Γ the Euler gamma function.

Definition (1.3.1),[14]:

A real function $f(x), x>0$, is said to be in the space $C_\mu, \mu\in R$, if there exists a real number p> μ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in c[0,1]$, Clearly $C_\mu\subset C_\beta$ if $\beta\leq\mu$.

Definition (1.3.2),[48]:

A real function $f(x), x>0,$ is said to be in the space C_μ^m,m$\in N\cup\{0\}$, if $f^{(m)}\in C_\mu$.

Definition (1.3.3),[14]:

The left and right Riemann-Liouville fractional integral of order $\alpha\geq 0$ of a function $f\in C_\mu, \mu\geq -1, is$ defined as follows:

$${}_aI_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

and ${}_xI_b^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$, respectively.

Definition (1.3.4),[9]:

The left and right Riemann-Liouville fractional derivatives of order α are defined by:

$${}_aD_x^\alpha f(x)=\frac{d^n}{dx^n}\,{}_aI_x^{n-\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\cdot\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha-1}f(t)\,dt$$

and

$${}_{x}D_{b}^{\alpha}f(x) = (-1)^{n}\frac{d^{n}}{dx^{n}}\,{}_{x}I_{b}^{n-\alpha}f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\cdot\frac{d^{n}}{dx^{n}}\int_{a}^{x}(t-x)^{n-\alpha-1}f(t)\,dt,$$

respectively.

Definition (1.3.5),[14]:

Let $f \in C_{-1}^{m}, m \in \mathbb{N} \cup \{0\}$.The Caputo fractional derivatives of $f(x)$ is defined by:

$${}_{a}^{c}D_{x}^{\alpha}f(x) = {}_{a}I_{x}^{n-\alpha}\frac{d^{n}}{dx^{n}}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}f^{(n)}(t)\,dt$$

and

$${}_{x}^{c}D_{b}^{\alpha}f(x) = (-1)^{n}\,{}_{x}I_{b}^{n-\alpha}\frac{d^{n}}{dx^{n}}f(x)$$

$$= \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(-1)^{n}(t-x)^{n-\alpha-1}f^{(n)}(t)\,dt.$$

respectively, where n is a natural number such that $n-1 \le \alpha < n$.

so, for $n = 1$ then $0 \le \alpha < 1$, the relations above take the following forms:

$${}_{a}^{c}D_{x}^{\alpha}f(x) = {}_{a}I_{x}^{1-\alpha}\frac{d}{dx}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{x}(x-t)^{-\alpha}\frac{d}{dx}f(t)\,dt \qquad (1.5a)$$

$${}_{x}^{c}D_{b}^{\alpha}f(x) = -{}_{x}I_{b}^{1-\alpha}\frac{d}{dx}f(x) = \frac{-1}{\Gamma(1-\alpha)}\int_{a}^{x}(t-x)^{-\alpha}\frac{d}{dx}f(t)\,dt. \qquad (1.5b)$$

Example (1.3.3),[32]:

The left hand Caputo fractional derivative of order $0 \le \alpha < 1$ for the function f defined by $f(t) = t^{b}$, b> 0 is

$${}_{a}^{c}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}(t-s)^{-\alpha}\frac{d}{ds}s^{b}\,dt$$

Hence, let s=t v , which implies ds=t dv in the above equation to get:

$${}_{a}^{c}D_{t}^{\alpha}f(t) = \frac{b}{\Gamma(1-\alpha)}\int_{0}^{1}(t-t\,v)^{-\alpha}(t\,v)^{b-1}t\,dv.$$

$$=\frac{b}{\Gamma(1-\alpha)}\int_0^1 t^{\,-\alpha}(1-v)^{-\alpha}(t\,)^{b-1}(v\,)^{b-1}t\,dv.$$

$$=\frac{b}{\Gamma(1-\alpha)}t^{b-\alpha}\int_0^1 (v\,)^{b-1}(1-v)^{-\alpha}\,dv.$$

$$=\frac{b}{\Gamma(1-\alpha)}t^{b-\alpha}\beta(b,1-\alpha)$$

But $b\,\Gamma(b)=\Gamma(b+1)$ therefore:

$${}_a^cD_t^\alpha f(t)=\frac{\Gamma(b+1)}{\Gamma(b+1-\alpha)}t^{b-\alpha}$$

The following properties are presented in details which will be needed later on :

1.The relation between Riemann-Liouville and Caputo fractional derivatives:

$${}_a^cD_x^\alpha f(x)={}_aD_x^\alpha f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}$$, and

$${}_x^cD_b^\alpha f(x)={}_xD_b^\alpha f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}$$

Therefore, if

$f(a)=\grave{f}(a)=\cdots=f^{(n-1)}(a)=0$ then ${}_a^cD_x^\alpha f(x)={}_aD_x^\alpha f(x)$

and, if

$f(b)=\grave{f}(b)=\cdots=f^{(n-1)}(b)=0$ then ${}_x^cD_b^\alpha f(x)={}_xD_b^\alpha f(x)$.[3]

2. Caputo fractional differential have some properties.

i. The fractional operators are linear,

$$P\big(\mu f(x)+\nu g(x)\big)=\mu Pf(x)+\nu Pg(x)$$

where P is a one of ${}_aD_x^\alpha$, ${}_xD_b^\alpha$, ${}_a^cD_x^\alpha$, ${}_x^cD_b^\alpha$, ${}_aI_x^\alpha$ *or* ${}_xI_b^\alpha$ and μ and ν are real numbers. [3]

ii. ${}_a^cD_x^\alpha \mathrm{C}=0$,where C is a constant. [20],[21] (1.6)

iii. $${}_a^cD_x^\alpha x^n\begin{cases}0 & \text{for } n\in\mathbb{N}_0 \text{ and } n<\alpha\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha} & \text{for } n\in\mathbb{N}_0 \text{ and } n\geq\alpha\end{cases}$$ [20]

(1.7).

3. If $f \in L_\infty(a,b)$ or $f \in C[a,b]$, and if $\alpha > 0$ then

$${}_a^cD_x^\alpha\, {}_aI_x^\alpha f(x) = f(x) \;\; and \;\; {}_x^cD_b^\alpha\, {}_xI_b^\alpha f(x) = f(x)$$

4. If $f \in C^n[a,b]$, and if $\alpha > 0$ then

$${}_aI_x^\alpha\, {}_a^cD_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, \text{ and}$$

$${}_xI_b^\alpha\, {}_x^cD_b^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k \;\; . [3]$$

5. Let $\alpha \geq 0$, $n = \lceil \alpha \rceil$ and $f(x) = (x-a)^c$ for some ≥ 0 , then

$${}_a^cD_x^\alpha f(x) = \begin{cases} 0 & if\ c \in \{0,1,2,\cdots,n-1\} \\ \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)}(x-a)^{c-\alpha} & if\ c \in \mathbb{N}\ and\ c \geq n \\ & or\ c \notin \mathbb{N}\ and\ c > n-1 \end{cases} \quad , \quad (1.8)$$

[9]

6. If *f* is a function such that *f(a)=f(b)=0, we have simpler* formulas

$$\int_a^b g(x).\, {}_a^cD_x^\alpha f(x).\, dx = \int_a^b f(x).\; {}_xD_b^\alpha . g(x).\, dx$$

and

$$\int_a^b g(x).\, {}_x^cD_b^\alpha f(x).\, dx = \int_a^b f(x).\; {}_aD_x^\alpha . g(x).\, dx \quad . [3]$$

7. Let $\geq 0\ and\ n = \lceil \alpha \rceil$. assume that f is such that both ${}_a^cD_x^\alpha f(x)$ exist then,

$${}_a^cD_x^\alpha f(x) = D_a^\alpha f(x) - \sum_{k=0}^{n-1} \frac{D^k f(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \;\; .[9]$$

8.The initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations.[3]

1.4 The Chebyshev Polynomials ,[29]:

Chebyshev polynomials take a significant position in modern developments including orthogonal polynomial, polynomial approximation, numerical integration, and spectral methods for partial differential equations. There are several kinds of Chebyshev polynomials in particular we shall introduce the first and second kind $T_n(x)$ and $U_n(x)$, as well as polynomials $V_n(x)$ and $W_n(x)$, which call the

Chebyshev polynomials of third and fourth kind's. In addition we cover the shifted polynomials $T_n^*(x)$, $U_n^*(x)$, $V_n^*(x)$ and $W_n^*(x)$.

Definition(1.4.6),[29]:

An integrable function ω is called a weight function on the interval I if $\omega(x) \geq 0$ for all x in I, but $\omega(x) \neq 0$ on any subinterval of I.

1.4.1 The Chebyshev Polynomial of First Kind,[44]:

It is well known that first kind chebyshev polynomial T_n(z) of degree n ,which defined on [-1,1] by :

T_n(z) = Cos(n θ) where z= Cos(θ) , where θ ∈ [0,π]

and can be determined with the aid of the following recurrence formula:

$$T_{n+1}(z)= 2z\, T_n(z)\, (z) - T_{n-1}(z) \qquad n=1,2,3,\ldots.. \qquad (1.9)$$

T_0(z)=1 , T_1(z) =z

The analytic form of Chebyshev polynomial T_n(x) of degree (n) is given by : $T_n(z) = \sum_{i=0}^{n/2} (-1)^i \; 2^{n-2i-1} \; \frac{n(n-i-1)!}{i!(n-2!)!} \; z^{n-2i}$ n=2,3..

and are orthogonal on [-1,1] with respect to the weight function $\omega_1(z)= 1/\sqrt{1-z^2}$, that is :

$$\int_{-1}^{1} \omega_1(z)\mathrm{T_n}\,(z).\,\mathrm{T_m}\,(z)\,dz = \begin{cases} \pi & n=m=0 \\ \frac{\pi}{2} & n=m\neq 0 \\ 0 & n\neq m \end{cases} \qquad (1.10)$$

1.4.2 The Shifted Chebyshev polynomial of First Kind,[29]:

In order to use these polynomials on the interval [0,1] , we define the shifted Chebyshev polynomials by introducing the change variable z=2x-1 , then T_n^*(x) can be obtained as follows :

$$\mathrm{T}_{n+1}^*(x) = 2(2x-1)\; \mathrm{T}_n^*(x) - \mathrm{T}_{n-1}\,(x) \qquad i=1,2,\ldots \qquad (1.11)$$

where, T_0^*(x)=1, T_1^*(x) =2x-1

and are orthogonal with respect to the weight function

$\omega_1^*(x)= 1/\sqrt{x-x^2}$, that is:

$$\int_0^1 \omega_1^*(x) T_n^*(x) . T_m^*(x)\, dx = \begin{cases} \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \\ 0 & n \neq m \end{cases} \quad (1.12)$$

Remark(1.4.1):

i. we drive the general analytic form which explained as:

$$T_n^*(x) = \sum_{i=0}^{n} (-1)^i \; 2^{2n-2i} \; \frac{n(2n-i-1)!}{i!(2n-2!)!} x^{n-i}, \quad n=2,3,\ldots$$

ii. $T_n^*(x)=n \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)!2^{2k}}{(n-k)!(2k)!} x^k$, n=2,3,… ,[4].

Remark(1.4.2),[29]:

It may define Chebyshev polynomials appropriate to any given finite range $[a\,,b]$ of x, by making this range correspond to the range $[-1\,,1]$ of a new variable s under the linear transformation

$$s = \frac{2x-(a+b)}{b-a} \quad (1.13)$$

The Chebyshev polynomials of the first kind appropriate to $[a\,,b]$ are thus $T_n(s)$, where (s) is given by (1.13) .

Example(1.4.4),[29]:

The first kind Chebyshev polynomial of degree three appropriate to the range $[1\,,4]$ of x is

$$T_3\left(\frac{2x-5}{3}\right) = 4\left(\frac{2x-5}{3}\right)^3 - 3\left(\frac{2x-5}{3}\right) = \frac{1}{27}(32x^3 - 240x^2 + 546x - 365)$$

Note that in the special case $[a, b] \equiv [0,1]$, the transformation (1.9) becomes s=2x-1.

1.4.3 The Shifted Pseudo-Spectral Chebyshev Polynomial of First Kind:

In order to use these polynomial on the interval x ∈ [0 , L] we defined the so called shifted pseudo-spectral Chebyshev polynomial of first kind by introducing the change of variable z= $\frac{2x}{L}$ -1. thus, $T_i(\frac{2x}{L} - 1)$, i=1,2,… denoted by T_i^p(x) Then T_i^p(x) can be obtained as follows :

$$T_{i+1}^p (x) = 2 (\frac{2x}{L} -1)T_i^p(x)- T_{i-1}^p (x), \quad i=1,2,\ldots \tag{1.14a}$$

where T_0^p(x)=1 and $T_1^p (x)=\frac{2x}{L} - 1$. (1.14b)

The analytic form of the shifted pseudo-spectral Chebyshev polynomial of first kind T_i^p(x) of degree i is given by :

$$T_i^p (x) = \sum_{i=0}^{n} (-1)^i \ 2^{2n-2i} \frac{n(2n-i-1)!}{i!(2n-2i)!\, L^{n-i}} x^{n-i} \tag{1.15}$$

Note that T_i^p (0) = $(-1)^i$ and T_i^p (L) =1 , the orthogonality condition is

$$\int_0^L T_i^p(x) \ . T_j^p(x) \ . \omega_1^p(x)\, dx = \sigma_j \tag{1.16}$$

where, $\sigma_j = \begin{cases} \pi & i = j = 0 \\ \frac{\pi}{2} & i = j \neq 0 \\ 0 & i \neq j \end{cases}$ and, $\omega_1^p(x) = \frac{1}{\sqrt{Lx-x^2}}$.

A function *u(x)*, square integrable in [0,L] , may be expressed in terms of shifted Chebyshev polynomials as,

$$u(x) = \sum_{j=0}^{\infty} b_j \ T_i^p(x) \tag{1.17}$$

where the coefficients b_j are given by ,

$$b_j=\frac{1}{\sigma_j} \int_0^L u(x)\, T_j^p(x).\omega_1^p(x)dx , \ j=0,1,2,\ldots\ldots \tag{1.18}$$

Theorem(1.4.1):

Let $y(x)$ be approximated by Chebyshev polynomials as

y_m(x) = $\sum_{i=0}^{m} c_i\, T_i^p(x)$ and also suppose α > 0 then,

$$D^\alpha(y_m(x))=\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i\ w_{i,k^{(\alpha)}}.x^{i-k-\alpha} \qquad (1.19)$$

where $w_{i,k}^{(\alpha)}$ is given by ,

$$w_{i,k}^{(\alpha)}=(-1)^k.2^{2i-2k}\frac{i\,(2\,i-k-1\,)\,!\ (\,i-k\,)!}{k\,!(\,2\,i-2\,k\,)!\ \Gamma(i-k+1-\alpha)\,L^{i-k}} \qquad (1.20)$$

<u>Proof</u>:

since the Caputo's fractional differentiation is a linear operation we have

$$D^\alpha(y_m(x))=\sum_{i=0}^{m} c_i.D^\alpha(T_i^p(x)) \qquad (1.21)$$

Where, $T_i^p(x)$=[$T_1^p(x)$, $T_2^p(x)$,….., $T_m^p(x)$]

Employing equation (1.6) and (1.7), we have

$$D^\alpha T_i^p(x)=0,\quad i=0,1,2,\dots\dots\dots\lceil\alpha\rceil-1,\quad \alpha>0 \qquad (1.22)$$

also, for i= $\lceil\alpha\rceil$,……,m , by using (1.7) , we get

$$D^\alpha\ (T_i^p\ (x)) = \sum_{k=0}^{i}(-1)^k\ 2^{2i-2k}\ \frac{i\,(\,2i-k-1\,)!}{(k)!\ \ (2i-2k\,)!\,\llcorner^{i-k}}\,D^\alpha\ \ x^{i-k}$$

$$=\sum_{k=0}^{i-\lceil\alpha\rceil}(-1)^k\ 2^{2i-2k}\ \frac{i\,(\,2i-k-1\,)!\,\Gamma(i-k+1)}{(k)!\ \ (2i-2k\,)!\,\Gamma(i-k+1-\alpha)\llcorner^{i-k}}\,x^{i-k-\alpha}$$

(1.23)

A combination of equations (1.21) , (1.22) and (1.23) leads to the desired result .

<u>1.4.4 Solution of Multi-Order Fractional Differential Equations</u> :

Consider the multi-order fractional differential equations of type given in (1.23)

$D^{\alpha}u(x)=F(x,u(x),D^{\beta 1}u(x),\ldots,D^{\beta m}u(x))$, $x\in[0,L]$ (1.24)

with initial conditions,

$u^{(k)}(0) = d_k$, $k = 0,1,\ldots\ldots,n$ (1.25)

where, $n < \alpha < n+1 = \lceil\alpha\rceil$, $0 < \beta_1 < \beta_2 < \ldots\ldots < \beta_m < \alpha$.

By the shifted Chebyshev pseudo-spectral polynomial of first kind as,

$$u(x) = \sum_{i=0}^{m} c_i\, T_i^p(x) = C^T\, T_i^p(x) \quad (1.26)$$

where, $T_i^p(x)=[\,T_1^p(x), T_2^p(x),\ldots, T_m^p(x)]$

$$D^{\alpha}u(x)=c^T D^{\alpha} T_i^p(x) \quad (1.27)$$

$$D^{sj}u(x)=c^T D^{sj} T_i^p(x) \quad (1.28)$$

where vector $c=[c_0, c_1, \ldots, c_m]^T$ an unknown vector .

By substituting these equations in equation (1.25) we obtain

$$u(0) = c^T\, T_i^p(0) = d_0$$

$$u^{(i)}(0)=c^T T_i^p(0)=d_k\ ,\quad k=0,1,\ldots\ldots\ldots,n \quad (1.29)$$

In order to use Chebyshev pseudo-spectral , we first approximate u(x) as,

$$u_i(x) = \sum_{i=0}^{m} u_i\, T_i^p(x) \quad (1.30)$$

From (1.24), (1.30), and theorem(1.4.1), we have

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\, w_{i,k}^{(\alpha)}\, x^{i-k-\alpha} = f\Big(x, \sum_{i=0}^{m} c_i T_i^p(x), \ldots, \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha 1\rceil} c_i w_{i,k}^{(\alpha 1)}\, x^{i-k-\alpha 1}, \sum_{i=\lceil\alpha m\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha m\rceil} c_i w_{i,k}^{(\alpha m)}\, x^{i-k-\alpha m 1}\Big)$$

(1.31)

we now collocate (1.31) at m+1-$\lceil\alpha\rceil$ points x_p as,

$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\, w_{i,k}^{(\alpha)}(x)\, x^{i-k-\alpha}$=$f(x_p$

$\sum_{i=0}^{m} c_i T_i^{p}(x_p), \dots, \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha 1\rceil} c_i w_{i,k}^{(\alpha 1)}(x_p)\, {x_p}^{i-k-\alpha 1}, \sum_{i=\lceil\alpha m\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha m\rceil} c_i w_{i,k}^{(\alpha m)}(x_p)\, {x_p}^{i-k-\alpha m})$

(1.32)

p= 0,1,……..,m-$\lceil\alpha\rceil$

For suitable collection point x_p we use m+1-$\lceil\alpha\rceil$ root of shifted chebyshev pseudo- spectral polynomial of first kind $T_{m+1-\lceil\alpha\rceil}^{p}(x)$.

also , by substituting (1.30) in the in initial condition (1.25) we can obtain $\lceil\alpha\rceil$ equations as follows,

$\sum_{i=0}^{m} u_i\, T_i^{p(k)}(0)=d_k$, k=0,1,……..,n (1.33)

m+1-$\lceil\alpha\rceil$ equations in (1.32) together with $\lceil\alpha\rceil$ equations of boundary condition (1.33) , generate (m+1) equations which can be solved by using newton's iterative method for the unknown u_i , i=0,1,………,m . consequently u(x) can be calculated .

Example(1.4.5):

Consider the following nonlinear differential equation

$D^3u(x) + D^{5/2}u(x) + u^2(x) = x^4$

$u(0) = u'(0) = 0,\ u''(0) = 2$

The exact solution is $u(x) = x^2$, [44].

By using equation (1.29), we get

u(0) = $c_0 - c_1 + c_2 - c_3 = 0$

u'(0) = $\frac{2}{L}\, c_1$ - $\frac{8}{L}\, c_2$ + $\frac{18}{L}\, c_3$ =0

$u''(0)$ = $\frac{16}{L^2}\, c_2$ - $\frac{96}{L^2}\, c_3$ =2

the systems of equation have , three equations with four unknowns and

$Ax = y \neq 0$

By using the shifted Chebyshev pseudo-spectral polynomial of first kind with m = 3 to obtain 3 .

$c_0 = \frac{3L^2}{8}$, $c_1 = \frac{L^2}{2}$, $c_2 = \frac{L^2}{8}$, $c_3 = 0$

then the approximate solution will be

$$\sum_{i=0}^{m} u_i\, T_i^p\,(x) = \frac{3L^2}{8} + \frac{L^2}{2}\left(\frac{2x}{L} - 1\right) + \frac{L^2}{8}\left(\frac{8x^2}{L^2} - \frac{8x}{L} + 1\right)$$

$$= \frac{3L^2}{8} + xL - \frac{L^2}{2} + x^2 - xL + \frac{L^2}{8} \cong x^2$$

It is clear that the approximate solution coincides with the analytic solution .

<u>1.4.5 The Chebyshev Polynomial of Second Kind,[29]</u>:

The second kind of degree (n),which defined on the interval $[-1\,,1]$ as:

$U_n(z) = \frac{\sin(n+1)\theta}{\sin\theta}$, where $z = \cos\theta$, $\theta \neq n\pi + 2k\pi$

These polynomials satisfy the following recurrence relation:

$$U_n(z) = 2zU_{n-1}(z) - U_{n-2}(z) \qquad n = 2\,,3\,,\cdots \tag{1.34}$$

where, $U_0(z) = 1, U_1(z) = 2z$.From Rodrigues formula

$$U_n(z) = \frac{(-2)^n(n+1)!}{(2n+1)!\sqrt{1-z^2}} D^n[(1-z^2)^{n+(\frac{1}{2})}]$$

and are orthogonal on [-1,1] with respect to the weight function

$\omega_2(z) = \sqrt{1-z^2}$, that is:

$$\int_{-1}^{1} \omega_2(z)U_m(z)U_n(z)dz = \begin{cases} 0 & m \neq n, \\ \frac{\pi}{2} & m = n \end{cases} \qquad (1.35)$$

1.4.6 The Shifted Chebyshev Polynomial of Second Kind,[7]:

The shifted Chebyshev polynomials of second kind are defined on interval [0,1] by introducing the change variable $z = 2x - 1$, then $U_n^*(x)$ can be obtained as follows:

$$U_n^*(x) = (4x - 2)U_{n-1}^*(x) - U_{n-2}^*(x) \qquad n = 2, 3, \cdots \qquad (1.36)$$

where, $U_0^*(x) = 1$, $U_1^*(x) = 4x - 2$

and the analytic form of shifted Chebyshev polynomials $U_n^*(x)$ of degree (n) is given by: $U_n^*(x) = \sum_{r=0}^{n+1} r\,(-1)^{n+1-r} \frac{(n+r)!\,2^{2r-1}}{(n+1-r)!\;\;2r!} x^{r-1}$

and are orthogonal with respect to the weight function

$\omega_2^*(x)=\sqrt{x-x^2}$, that is:

$$\int_0^1 \omega_2^*(x)U_m^*(x) \cdot U_n^*(x)\,dx = \begin{cases} \frac{\pi}{8} & m = n \\ 0 & m \neq n \end{cases} \qquad (1.37)$$

1.4.7 The Chebyshev Polynomials of Third and Fourth Kinds:

The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are polynomials of degree n in x defined respectively in [1],[29] by

$$V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta} \quad \text{, and} \quad W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}$$

where $= \cos\theta$, they may be generated by using the two recurrence relations $V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x) \qquad n = 2,3,\cdots \qquad (1.38)$

with the initial values $V_0(x) = 1$, $V_1(x) = 2x - 1$.

also,

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x) \qquad n = 2,3,\cdots \qquad (1.39)$$

with the initial values $W_0(x) = 1$, $W_1(x) = 2x + 1$,

and are orthogonal on [-1,1] , that is

$$\int_{-1}^{1} \omega_3(x)V_m(x)V_n(x)dx = \int_{-1}^{1} \omega_4(x)W_m(x)W_n(x)dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases}$$

(1.40)

where, $\omega_3(x) = \sqrt{\frac{1+x}{1-x}}$, $\omega_4(x) = \sqrt{\frac{1-x}{1+x}}$ are weight function of third and fourth respectively.

1.4.8 The Shifted Chebyshev Polynomials of Third and Fourth Kinds:

The shifted Chebyshev polynomials of the third kind is defined on [0,1], respectively in [29],[36].

$$V_n^*(x) = V_n(2x-1)\ . \tag{1.41a}$$

and satisfies to the following recurrence formula:

$$V_{n+1}^*(x) = 2(2x-1)V_n^*(x) - V_{n-1}^*(x), n = 1,2,\dots \tag{1.41b}$$

with the initial values

$$V_n^*(x) = 1\ ,\ V_1^*(x) = 4x-3 \tag{1.41c}$$

also, the shifted Chebyshev polynomial of fourth kind is defined on [0,1],

$$W_n^*(x) = W_n(2x-1). \tag{1.42a}$$

and satisfy to the following recurrence formula:

$$W_{n+1}^*(x) = 2(2x-1)W_n^*(x) - W_{n-1}^*(x),\ \ n = 1,2,\dots \tag{1.42b}$$

with the initial values

$$W_n^*(x) = 1, W_1^*(x) = 2x+1 \tag{1.42c}$$

The orthogonality relations of $V_n^*(x)$ and $W_n^*(x)$ on [0,1] are given by

$$\int_0^1 \omega_3^*(x)V_n^*(x)V_m^*(x)dx = \int_0^1 \omega_4^*(x)W_n^*(x)W_m^*(x)dx = \begin{cases} \frac{\pi}{2} & \mathrm{m = n} \\ 0 & \mathrm{m \neq n} \end{cases}$$

(1.43)

where $\omega_3^*(x) = \sqrt{\frac{x}{1-x}}$, $\omega_4^*(x) = \sqrt{\frac{1-x}{x}}$ are weight function of shifted Chebyshev polynomials of third and fourth respectively.

1.4.9 Connections Between the Four Kinds of polynomial,[29]:

1. The relationship between the polynomials T_n , U_n are

$U_n(x) - U_{n-2}(x) = 2T_n(x)$,n=2,3,… (1.44)

2. The relationship between the polynomials T_n , U_n and V_n , W_n needed two auxiliary variables:

$$u = \left(\frac{1}{2}(1+x)\right)^{\frac{1}{2}} = \cos\frac{1}{2}\theta, x \in [-1,1]. \quad (1.45)$$

$$t = \left(\frac{1}{2}(1-x)\right)^{\frac{1}{2}} = \sin\frac{1}{2}\theta, x \in [-1,1]. \quad (1.46)$$

(i) $T_n(x) = T_{2n+1}(u)$, and $U_n(x) = \frac{1}{2}U_{2n+1}(u)$.

(ii) $V_n(x) = U^{-1}T_{2n+1}(u)$, and $W_n(x) = U_{2n}(u)$.

(iii) $U_n(x) = \frac{1}{2}[V_n(x) + W_n(x)]$, (1.47a)

$V_n(x) = U_n(x) - U_{n-1}(x)$, (1.47b)

$W_n(x) = U_n(x) + U_{n-1}(x)$ (1.47c)

(iv) $W_n(x) = V_n(-x)$, $(n\ even)$

$W_n(x) = -V_n(-x)$, $(n\ odd)$.

<u>1.4.10 Some Relations Between Polynomials and There's Shifted,[29]:</u>

1. $T_{2n-1}(x) = xV^*_{n-1}(x^2)$.

2. $U_{2n-1}(x) = 2xU^*_{n-1}(x^2)$.

3. $U_{2n}(x) = W^*_n(x^2)$.

4. From (1.25) , we have

$W^*_{n-1}(u^2) = W_{n-1}(2u^2-1) = W_{n-1}(x) = U_{2n}(u)$.

<u>1.5 Wavelet Functions,[16]:</u>

Wavelets constitute a family of function constructed from dilations and translations of a single function called the mother wavelet $\psi(t)$. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\psi\left(\frac{t-b}{a}\right) \qquad a, b \in R , a \neq 0$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$,

$b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, for n and k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a|^k \psi\left(a_0^{-k} t - nb_0\right)$$

Where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular when $a_0 = 2$ and $b_0 = 1$, $\psi_{k,n}(t)$ forms an orthonormal basis this is

$< \psi_{k,n}(t), \psi_{l,m}(t) >= \mathcal{S}_{kl}\mathcal{S}_{nm}$, where, $\mathcal{S}_{kl}$ is the kronecker function .

<u>1.5.1 The Chebyshev Wavelets of First Kind,[16]:</u>

The shifted Chebyshev wavelets of first kind $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments, $k \in N$, $n = 1,2,\cdots,2^{k-1}$ and $\hat{n} = 2n - 1$ moreover, m is the order of the Chebyshev polynomials of firs kind and t is the normalized time, and they are defined on the interval $[0\ ,1)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \cdot T_m^*(2^k t - \hat{n}) & \frac{\hat{n}-1}{2^k} \le t < \frac{\hat{n}}{2^k} \\ 0 & otherwise \end{cases} \qquad (1.48a)$$

$$\text{Where } T_m^*(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0 \\ \sqrt{\frac{2}{\pi}}\ T_m(t) & m > 0 \end{cases} \qquad (1.48b)$$

where $m = 0,1,\cdots,M-1$ and M is a fixed positive integer. The coefficients in (1.28b) are used for orthonomality.

with the weight function $\omega_1^* = \omega_1(2t-1)$ has to be dilated and translated as follows:

$$\omega_{1,n}^*(t) = \omega_1(2^k t - \hat{n})$$

and $\omega_2(2t-1)$ *has to be dilated and translated as follows,*

$$\omega_{1,n}^*(2^k\ t\ \text{-}n) = \frac{1}{\sqrt{(2^k\ t - n) - (2^k\ t - n)^2}} .$$

<u>1.5.2 The Chebyshev Wavelets of Second Kind,[2]:</u>

Chebyshev wavelets of second kind $\Psi^2_{nm}(t) = \Psi^2$ (k,n,m,t) have four arguments k , n can assume any positive integer , m is the order Chebyshev polynomials of second kind, and t is the normalized time. They are defined on the interval [0,1] by,

$$\Psi^2_{nm}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m(2^k t - n) & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}] \\ 0 & o.w \end{cases} \quad (1.49)$$

m=0,1,……M, *n=0,1,……* 2^k*-1*

with the weight function $\omega^*_2 = \omega_2(2t-1)$ has to be dilated and translated as follows:

$$\omega^*_{2,n}(t) = \omega_2(2^k t - \hat{n})$$

and $\omega_2(2t-1)$ *has to be dilated and translated as follows,*

$$\omega^*_{2,n}(2^k t - n) = \sqrt{(2^k t - n) - (2^k t - n)^2}$$

<u>1.5.3 Function Approximation,[16]:</u>

A function $f(t)$ defined over $[0, 1)$ may be expanded as follows:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t) \quad (1.50)$$

By the shifted Chebyshev wavelets of first kind, where

$$c_{n,m} = \left(f(t), \psi_{n,m}(t)\right)$$

If the infinite series in (1.29) is truncated, then (1.29) can be written as

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = c^T \psi(t) \quad (1.51)$$

where, c and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$c = \left[c_{10}, c_{11}, \cdots, c_{1M-1}, c_{20}, \cdots, c_{2M-1}, \cdots, c_{2^{k-1}0}, \cdots, c_{2^{k-1}M-1}\right]^T \quad (1.52)$$

$$\psi = \left[\psi_{10}(t), \psi_{11}(t), .., \psi_{1M-1}(t), \psi_{20}(t), .., \psi_{2M-1}(t), .., \psi_{2^{k-1}0}(t), .., \psi_{2^{k-1}M-1}(t)\right]^T.$$

Taking the collocation points $t_i = \frac{(2i-1)}{2^k M}$ $\quad i = 1,2, \cdots, M$

where $m = 2^{k-1}M$, we define the wavelet matrix $\emptyset_{m \times m}$ as

$$\emptyset_{m\times m} = \left[\psi\left(\frac{1}{2m}\right), \psi\left(\frac{3}{2m}\right), \cdots, \psi\left(\frac{2m-1}{2m}\right)\right] \tag{1.53}$$

Indeed $\emptyset_{m\times m}$ has the following form:

$$\emptyset_{m\times m} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{bmatrix} \tag{1.54}$$

where A is $M \times M$ matrix given by

$$A = \begin{bmatrix} \psi_{10}(\frac{1}{2m}) & \psi_{10}(\frac{3}{2m}) & \cdots & \psi_{10}(\frac{2m-1}{2m}) \\ \psi_{11}(\frac{1}{2m}) & \psi_{11}(\frac{3}{2m}) & \cdots & \psi_{11}(\frac{2m-1}{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{1M-1}(\frac{1}{2m}) & \psi_{1M-1}(\frac{3}{2m}) & \cdots & \psi_{1M-1}(\frac{2m-1}{2m}) \end{bmatrix} \tag{1.55}$$

For example, for $M = 4$ and $k = 2$, shifted chebyshev matrices of first kind can be expressed as $\emptyset_{m\times m} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$

and for the chebyshev matrix we have

$$A = \begin{bmatrix} 1.12838 & 1.12838 & 1.12838 & 1.12838 \\ -1.19683 & -0.398942 & -0.398942 & 1.19683 \\ 0.199471 & -1.39630 & -1.39630 & 0.199471 \\ 0.897621 & 1.09709 & -1.09709 & -0.897621 \end{bmatrix}$$

for example $k = 1$ and $M = 2$, we have the shifted chebyshev wavelet of first kind can be expressed as : $n = 0$, and $m = 0,1$

$$\left.\begin{aligned} \psi_{0,0}(t) &= 2^{\frac{1}{2}}\,\tilde{T}_0(2t+1) = \frac{\sqrt{2}}{\sqrt{\pi}} \\ \psi_{0,1}(t) &= 2^{\frac{1}{2}}\,\tilde{T}_1(2t+1) = \frac{2}{\sqrt{\pi}}(2t+1) \end{aligned}\right\} \quad -1 \le t < \frac{-1}{2}.$$

And for example $k = 0$ and M= 1 , we have The shifted chebyshev wavelet of second kind can be expressed as :

$n = 0$, and $m = 0,1$

$$\left.\begin{aligned}\psi_{0,0}(t) &= \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} U_0^*(t) = \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} \\ \psi_{0,1}(t) &= \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} U_1^*(t) = \frac{2^{\frac{3}{2}}}{\sqrt{\pi}}(4x-3)\end{aligned}\right\} \quad t \in [0,1]$$

1.5.4 Convergence Analysis:

Theorem(1.5.2),[18]:

A function $f(x)$ defined on $[0\,,1)$, is with bounded second derivative, say $\left|\grave{\grave{f}}(x)\right| \leq B$, can be expanded as an infinity sum of shifted Chebyshev wavelets of first kind, and the series converges uniformly to the function $f(x)$. Explicitly, the expansion coefficients in

$$c_{n,m} = \left(f(t), \psi_{n,m}(t)\right) = \int_0^1 \omega_1^*(t) \cdot f(t) \cdot \psi_{n,m}(t)\, dt$$

Satisfy the following inequality

$\left|c_{n,m}\right| \leq \frac{\sqrt{2\pi} \cdot B}{(2n)^{\frac{9}{2}}\,(m^2-1)}$.

Theorem(1.5.3),[2]:

A function $f(x) \in L_w^2[0\,,1]$, with $\left|\grave{\grave{f}}(x)\right| \leq L$, can be expanded as an infinity sum of shifted Chebyshev wavelets of second kind, and the series converges uniformly to the function $f(x)$. Explicitly, the expansion coefficients in $c_{n,m} = \left(f(t), \psi_{n,m}(t)\right) = \int_0^1 \omega_2^*(t) \cdot f(t) \cdot \psi_{n,m}(t)\, dt$

Satisfy the following inequality $\left|c_{n,m}\right| < \frac{8\sqrt{2\pi} \cdot L}{(n+1)^{\frac{5}{2}}(m+1)^2}$.

Theorem(1.5.4),[29]:

Assume that a function $f(x) \in L^2_{w_3^*}[0\,,1)$, $\omega_3^* = \sqrt{t/1-t}$ with $\left|\grave{\grave{f}}(x)\right| \leq L$, can be expanded as an infinity series of shifted Chebyshev wavelets of third kind, then this series converges uniformly to $f(x)$. Explicitly, the expansion coefficients in

$$c_{n,m} = \left(f(t), \psi_{n,m}(t)\right)_{w_i^*} = \int_0^1 \omega_3^*(t) \cdot f(t) \cdot \psi_{n,m}(t)\, dt$$

Satisfy the following inequality

$$\left|c_{n,m}\right| \le \frac{2\sqrt{2\pi} \cdot L \cdot m^2}{(n+1)^{\frac{5}{2}}\,(m^4-1)} \qquad \forall\, n \ge 0\,, m > 1\,.$$

The following theorem appears in [29], The proof is similar to that exist for ω_3^* in reference [29].

Theorem(1.5.5):

Assume that function $(x) \in L^2_{\omega_4^*}[0\,,1]$, $\omega_4^* = \sqrt{\frac{1-x}{x}}$ with $\left|\dddot{f}(x)\right| \le N$, can be expanded as an infinite series of fourth-kind Chebyshev wavelets ; then this series converges uniformly to $f(x)$. Explicitly, the expansion coefficients in

$$c_{n,m} = \left(f(t), \psi_{n,m}(t)\right)_{w_i^*} = \int_0^1 \omega_4^*(t) \cdot f(t) \cdot \psi_{n,m}(t)\, dt. \qquad (1.56)$$

satisfy the following inequality $|c_{nm}| < \frac{3.N.\sqrt{\pi^3}}{\sqrt{2} \cdot (n+1)^{\frac{3}{2}} \cdot k(k+1)}$

Proof:

From (1.35), it follows that

$$c_{nm} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t) \cdot W_m^*(2^k t - n) \cdot \omega_4^*(2^k t - n) dt.$$

Let $(2^k t - n) = \cos\theta$ then $t = \frac{\cos\theta + n}{2^k}$ yield $dt = \frac{-1}{2^k} \sin\theta\, d\theta$

$$2^k \cdot \frac{n+1}{2^k} - n = \cos\theta \quad \text{thus} \quad \theta = 0$$

$$2^k \cdot \frac{n}{2^k} - n = \cos\theta \quad \text{therefore} \quad \theta = \frac{\pi}{2}$$

$$c_{nm} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{\frac{\pi}{2}}^{0} f\left(\frac{\cos\theta + n}{2^k}\right) \cdot W_m^*(\cos\theta) \cdot \omega_4^*(\cos\theta) \cdot \frac{-1}{2^k} \sin\theta \, d\theta$$

$$= \frac{2^{\frac{1-k}{2}}}{\sqrt{\pi}} \int_{\frac{\pi}{2}}^{0} f\left(\frac{\cos\theta + n}{2^k}\right) \cdot W_m^*(\cos\theta) \cdot \omega_4^*(\cos\theta) \cdot -\sin\theta \, d\theta$$

since $W_m^*(\cos\theta) = W_m(2\cos\theta - 1)$

$$c_{nm} = \frac{2^{\frac{1-k}{2}}}{\sqrt{\pi}} \int_{\frac{\pi}{2}}^{0} f\left(\frac{\cos\theta + n}{2^k}\right) \cdot W_m^*(2\cos\theta - 1) \cdot \omega_4^*(\cos\theta) \cdot (-\sin\theta) \, d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1-k}{2}}} \int_0^{\pi} f\left(\frac{\cos\theta+n}{2^k}\right) \cdot W_m(x) \cdot \omega_4^*(\cos\theta) \cdot \sin\theta \, d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1-k}{2}}} \int_0^{\pi} f\left(\frac{\cos\theta+n}{2^k}\right) \cdot \frac{\sin\left(k+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} \cdot \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot \sqrt{1-\cos\theta} \,.\, \sqrt{1+\cos\theta} \, d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{-1-k}{2}}} \int_0^{\pi} f\left(\frac{\cos\theta+n}{2^k}\right) \cdot \sin\left(k+\frac{1}{2}\right)\theta \cdot \sin\frac{\theta}{2} \, d\theta \,.$$

Thus,

$$c_{nm} = \frac{\sqrt{\pi}}{2^{\frac{1-k}{2}}} \int_0^{\pi} f\left(\frac{\cos\theta + n}{2^k}\right) \cdot [\cos(k\theta) - \cos(k+1)\theta] \, d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1-k}{2}} . 2^k} \int_0^{\pi} \grave{f}\left(\frac{\cos\theta+n}{2^k}\right) \cdot \sin\theta . \cdot \left(\frac{\sin(k\theta)}{k} - \frac{\sin(k+1)\theta}{k+1}\right) d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1+k}{2}} . k} \int_0^{\pi} \grave{f}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \sin\theta \cdot \sin(k\theta) \, d\theta$$

$$- \frac{\sqrt{\pi}}{2^{\frac{1+k}{2}} . (k+1)} \int_0^{\pi} \grave{f}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \sin\theta \cdot \sin(k+1)\theta \, d\theta$$

1) $\int_0^{\pi} \grave{f}\left(\frac{\cos\theta+n}{2^k}\right) \cdot \sin(k\theta) \cdot \sin\theta \, d\theta$

$$= \int_0^{\pi} \grave{f}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \left[\frac{1}{2}\left(\cos(k-1)\,\theta - \quad \cos(k+1)\theta\right)\right] d\theta$$

$$= \grave{f}\left(\frac{\cos\theta + n}{2^k}\right)$$

$$\cdot \left[\frac{1}{2(k-1)}\sin(k-1)\theta - \frac{1}{2(k+1)}\sin(k+1)\theta\right]\Bigg|_{0}^{\pi}$$

$$+ \frac{1}{2^k \cdot 2}\int_0^{\pi} \grave{\grave{f}}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \sin\theta$$

$$\cdot \left[\frac{\sin(k-1)\theta}{k-1} - \frac{\sin(k+1)\theta}{k+1}\right] d\theta$$

2)$\int_0^{\pi} \grave{f}\left(\frac{\cos\theta+n}{2^k}\right) \cdot \sin\theta \cdot \sin(k+1)\theta \, d\theta$

$$= \int_0^{\pi} \grave{f}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \left[\frac{1}{2}\cos(k\theta) - \cos(k+2)\theta\right] d\theta$$

$$= \grave{f}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \left[\frac{\sin(k\theta)}{2k} - \frac{\sin(k+2)\theta}{2(k+2)}\right]\Bigg|_{0}^{\pi}$$

$$+ \frac{1}{2^k \cdot 2}\int_0^{\pi} \grave{\grave{f}}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \sin\theta$$

$$\cdot \left[\frac{\sin(k\theta)}{k} - \frac{\sin(k+2)\theta}{k+2}\right] d\theta$$

Thus,

$$c_{nm} = \frac{\sqrt{\pi}}{2^{\frac{3+3k}{2}}}\int_0^{\pi} \grave{\grave{f}}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \sin\theta$$

$$\cdot \left[\frac{1}{k}\left(\frac{\sin(k-1)\theta}{k-1} - \frac{\sin(k+1)\theta}{k+1}\right)\right.$$

$$\left. - \frac{1}{k+1}\left(\frac{\sin(k\theta)}{k} - \frac{\sin(k+2)\theta}{k+2}\right)\right] d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{3+3k}{2}}}\int_0^{\pi} \grave{\grave{f}}\left(\frac{\cos\theta + n}{2^k}\right) \cdot \left[\frac{1}{k} r_m(\theta) - \frac{1}{k+1} d_m(\theta)\right] d\theta$$

Since $r_m(\theta) = \sin\theta \cdot \left[\frac{\sin(k-1)\theta}{k-1} - \frac{\sin(k+1)\theta}{k+1}\right]$

$$d_m(\theta) = \sin\theta \cdot \left[\frac{\sin(k\theta)}{k} - \frac{\sin(k+2)\theta}{k+2}\right]$$

we get,

$$|c_{nm}| = \left|\frac{\sqrt{\pi}}{2^{\frac{3+3k}{2}}}\int_0^{\pi} \grave{\grave{f}}\left(\frac{\cos\theta + n}{2^k}\right)\cdot\left[\frac{1}{k}r_m(\theta) - \frac{1}{k+1}d_m(\theta)\right]d\theta\right|$$

$$\leq \frac{\sqrt{\pi}.N}{2^{\frac{3+3k}{2}}}.\int_0^{\pi}\left|\grave{\grave{f}}\left(\frac{\cos\theta + n}{2^k}\right)\right.$$

$$\left.\cdot\left[\frac{1}{k}r_m(\theta) - \frac{1}{k+1}d_m(\theta)\right]\right| d\theta$$

However,

$$\int_0^{\pi}\left|\frac{1}{k}r_m(\theta) - \frac{1}{k+1}d_m(\theta)\right| d\theta$$

$$\leq \frac{1}{k}\int_0^{\pi}\left|\sin\theta\cdot\left[\frac{\sin(k-1)\theta}{k-1} - \frac{\sin(k+1)\theta}{k+1}\right]\right| d\theta$$

$$-\frac{1}{k+1}\int_0^{\pi}\left|\sin\theta\cdot\left[\frac{\sin(k\theta)}{k} - \frac{\sin(k+2)\theta}{k+2}\right]\right| d\theta$$

From (1),(2), we get

$$c_{nm} < \frac{\sqrt{\pi}.N}{2^{\frac{3+3k}{2}}}\left|\frac{2\cdot\pi}{k(k^2-1)} - \frac{2\cdot\pi}{k(k+1)(k+2)}\right|$$

$$< \frac{3N\sqrt{\pi^3}}{2^{\frac{1+3k}{2}}\cdot k(k+1)}$$

Finally, since $n \leq 2^k - 1$ we have,

$$|c_{nm}| < \frac{3.N\cdot\sqrt{\pi^3}}{\sqrt{2}.(2^k)^{\frac{3}{2}}\cdot k(k+1)}$$

$$|c_{nm}| < \frac{3.N.\sqrt{\pi^3}}{\sqrt{2}\cdot(n+1)^{\frac{3}{2}}\cdot k(k+1)}$$

CHAPTER TWO

Operational Matrices of Fractional Derivative for Solving (MFNDE)

2.1 Introduction:

In some cases, the analytical solution may be difficult to evaluate , therefore numerical and approximate methods seem to be necessary to be used which cover the problem under consideration. The method that will be considered in this work is to find the operational matrices of fractional derivatives for different types of Chebyshev polynomials to find the solution of multi-fractional order nonlinear differential equation(MFNDE), also, the author method that will be considered in this work is the operational matrix of fractional derivatives of Chebyshev wavelets functions for solving the multi-fractional order nonlinear differential equation(MFNDE) .

The relation between the different types of wavelets Chebyshev has been given as a relation between the operational matrices of these types with their proving in details . So the general operational matrices are presented to support the filed in these projects.

This chapter consists of three sections. In section (2.2), the operational matrix of fractional derivative for Chebyshev polynomial of shifted first kind and shifted second kind is presented. In section (2.3), the operational matrix of fractional derivative for Chebyshev wavelets of shifted first, second, third and fourth kinds is given. Finally, in section (2.4), the new relation between operational of fractional derivative for Chebyshev wavelets of shifted second with third kinds, Chebyshev wavelets of shifted second with fourth kinds, chebyshev wavelets of shifted second, third and fourth kinds, Chebyshev wavelets for shifted first with second kind is presented.

2.2 The Operational Matrix of Fractional Derivative for Chebyshev polynomial:

The proposed operational matrix formulation of fractional order derivative $\alpha>0$ for shifted first and second kinds Chebyshev polynomial is denoted by $D^{\alpha}\emptyset(x), D^{\alpha}\varphi(x)$, respectively ,where $\emptyset(x) = [T_0^*(x), T_1^*(x), \ldots, T_n^*(x)]^T$ and $\varphi(x) = [U_0^*(x), U_1^*(x), \ldots, U_n^*(x)]^T$,together have been expressed in details.

Theorem(2.1.6):

Let $\emptyset(x)$ be shifted first kind Chebyshev vector defined as
$\emptyset(x) = [T_0^*(x), T_1^*(x), \ldots, T_n^*(x)]^T$ and also suppose $\alpha>0$ then

$D^{\alpha}\,\emptyset(x) = \Delta^{\alpha}\emptyset(x)$

where, Δ^{α} is (m+1)(m+1) is an operational matrix of fractional derivative of order $\alpha>0$ in the Caputo sense and is defined as follows,

$$\Delta^{\alpha} = \begin{bmatrix} 0 & & 0 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & & 0 & & 0 \\ w^{(1)}_{0,0,i} & & w^{(1)}_{0,1,i} & \cdots & w^{(1)}_{0,m1,i} \\ \vdots & & \vdots & & \vdots \\ \sum_{i=0}^{n-\lceil\alpha\rceil} w^{(1)}_{n-\lceil\alpha\rceil,0,i} & & \sum_{i=0}^{n-\lceil\alpha\rceil} w^{(1)}_{n-\lceil\alpha\rceil,1,i} & \cdots & \sum_{i=0}^{n-\lceil\alpha\rceil} w^{(1)}_{n-\lceil\alpha\rceil,m,i} \\ \vdots & & \vdots & & \vdots \\ \sum_{i=0}^{m} w^{(1)}_{m,0,i} & & \sum_{i=0}^{m} w^{(1)}_{m,1,i} & \cdots & \sum_{i=0}^{m} w^{(1)}_{m,m,i} \end{bmatrix} \tag{2.1}$$

and, $w^{(1)}_{n-\lceil\alpha\rceil,j,i}$ is given by

$$w^{(1)}_{n-\lceil\alpha\rceil,j,i} = \frac{\sigma_j}{\sqrt{\pi}} \sum_{k=0}^{j} (-1)^{k+i}\, 2^{2j-2k+2n-2i} \frac{n(2n-i-1)!\,j(2j-k-1)!\,\Gamma(n-i-\alpha+j-k+\frac{1}{2})}{i!(2n-2!)!\,\Gamma(n-i-\alpha+1)k!(2j-2k)!\,\Gamma(n-i-\alpha+j-k+1)} \tag{2.2}$$

where, $n=\lceil\alpha\rceil \ldots m$, and $\sigma_j = \begin{cases} 1 & j = 0 \\ 2 & j \neq 0 \end{cases}$

Proof :

Let $T_m^*(x)$ be shifted first kind Chebyshev polynomial then by using (1.7) we can find that,

$D^{\alpha}\, T_n^*(x) = 0$, $n < \lceil\alpha\rceil$, and for $n \geq \lceil\alpha\rceil$.

$$D^{\alpha} T_n^*(x) = \sum_{i=0}^{n} (-1)^i\, 2^{2n-2i} \frac{n(2n-i-1)!}{i!(2n-2!)!} D^{\alpha} x^{n-i}$$

$$= \sum_{i=0}^{n-\lceil\alpha\rceil} (-1)^i\, 2^{2n-2i} \frac{n(2n-i-1)!\Gamma(n-i+1)}{i!(2n-2!)!\,\Gamma(n-i-\alpha+1)} x^{n-i-\alpha}\ . \tag{2.3}$$

Now, approximate $x^{n-i-\alpha}$ by (m+1)-terms of shifted Chebyshev polynomial of first kind , we have

$$x^{n-i-\alpha} = \sum_{j=0}^{m} d_{n-i,j}\, T_j^*(x) \tag{2.4a}$$

where ,

$$d_{n-i,j} = \frac{\sigma_j}{\pi} \int_0^1 \frac{x^{n-i-\alpha}}{\sqrt{x-x^2}}\, T_j^*(x)\, dx \tag{2.4b}$$

$$T_j^*(x) = \sum_{k=0}^{j} (-1)^k\, 2^{2j-2k} \frac{j(2j-k-1)!}{k!(2j-2k)!} x^{j-k} \tag{2.4c}$$

then, $d_{n-i,j} = \frac{\sigma_j}{\pi} \sum_{k=0}^{j} (-1)^k\, 2^{2j-2k} \frac{j(2j-k-1)!}{k!(2j-2k)!} \int_0^1 \frac{x^{n-i-\alpha+j-k}}{\sqrt{x-x^2}} .dx$

$$= \frac{\sigma_j}{\pi} \sum_{k=0}^{j} (-1)^k\, 2^{2j-2k} \frac{j(2j-k-1)!\,\Gamma(n-i-\alpha+j-k+\frac{1}{2})\sqrt{\pi}}{k!(2j-2k)!\,\Gamma(n-i-\alpha+j-k+1)} \quad \text{where,} \tag{2.5}$$

$\sigma_j = \begin{cases} 1 & j = 0 \\ 2 & j \neq 0 \end{cases}$. By substituting (2.5) in (2.3), we get

$D^{\alpha}\ T_n^*(x) = \sum_{i=0}^{n-\lceil\alpha\rceil} \sum_{j=0}^{m} (-1)^i\ 2^{2n-2i}\ \frac{n(2n-i-1)!(n-i)!}{i!(2n-2!)!\ \Gamma(n-i-\alpha+1)}\ d_{n-i,j}\ \ T_n^*(x)$

$= \frac{\sigma_j}{\pi} \sum_{i=0}^{n-\lceil\alpha\rceil} \sum_{j=0}^{m} \sum_{k=0}^{j} (-1)^{k+i}\ 2^{2j-2k+2n-2i}\ \frac{n(2n-i-1)!j(2j-k-1)\ \Gamma\left(n-i-\alpha+j-k+\frac{1}{2}\right).(n-i)!}{i!(2n-2!)!\ \Gamma(n-i-\alpha+1)\ k!(2j-2k)!\ \Gamma(n-i-\alpha+j-k+1)}\ T_n^*(x)$

where,

$w_{n-\lceil\alpha\rceil,j,i}^{(1)} = \frac{\sigma_j}{\sqrt{\pi}} \sum_{k=0}^{j} (-1)^{k+i}\ 2^{2j-2k+2n-2i}\ \frac{n(2n-i-1)!j(2j-k-1)!\ \Gamma(n-i-\alpha+j-k+\frac{1}{2})}{i!(2n-2!)!\ \Gamma(n-i-\alpha+1)k!(2j-2k)!\ \ \Gamma(n-i-\alpha+j-k+1)}$

$D^{\alpha}\ T_n^*(x) = \sum_{j=0}^{m} [\ \sum_{i=0}^{n-\lceil\alpha\rceil} w_{n,j,i}^{(1)}\ \ ]\ T_n^*(x)$,for $n \geq \lceil\alpha\rceil$. (2.6a)

$= \left[\sum_{i=0}^{n-\lceil\alpha\rceil} w_{n-\lceil\alpha\rceil,0,i}^{(1)},\ \sum_{i=0}^{n-\lceil\alpha\rceil} w_{n-\lceil\alpha\rceil,1,i}^{(1)}, \ldots, \sum_{i=0}^{n-\lceil\alpha\rceil} w_{n-\lceil\alpha\rceil,m,i}^{(1)}\right] \emptyset(x)$

,for $n \geq \lceil\alpha\rceil$ also, $D^{\alpha}\ T_n^*(x) = [\ 0,\ldots\ldots0\ ]\ \emptyset(x)$, $n < \lceil\alpha\rceil$. (2.6b)

2.2.1 Function Approximation of Shifted Chebyshev polynomial of First Kind,[44]:

The function u(x) square integrable in [0,1] , may be expressed in the term of shifted first kind Chebyshev polynomial as:

$u(x) = \sum_{i=0}^{\infty} c_i\ T_i^*(x)$

where the coefficients c_i are given by in (1.12) .In practice , only the first (m+1)-terms shifted first kind Chebyshev polynomial are considered ,then we have $u_m(x) = \sum_{i=0}^{m} c_i\ T_i^*(x) = C^T\emptyset(x)$, where $\emptyset(x)=[T_0^*(x)\ , T_1^*(x)\ , \ldots, T_n^*(x)]^T$

Example(2.1.6):

Consider the following multi-fractional order nonlinear differential equation

$$D^4u(x) + D^{\frac{7}{2}}u(x) + u^3(x) = x^9 \quad (2.7)$$

subject to the initial conditions

$$u(0) = u^{(1)}(0) = u^{(2)}(0) = 0\ \ ,\ u^{(3)}(0) = 6\ \ ,\ [19] \quad (2.8)$$

To solve the above problem with m = 4, by equation (2.1) from theorem(2.1.6), we have that

$$\Delta^{\frac{7}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sum_{i=0}^{4} W_{4,0,i}^{(1)} & \sum_{i=0}^{4} W_{4,1,i}^{(1)} & \sum_{i=0}^{4} W_{4,2,i}^{(1)} & \sum_{i=0}^{4} W_{4,3,i}^{(1)} & \sum_{i=0}^{4} W_{4,4,i}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 61.299 & -12.26 & 5.254 & -2.919 \end{bmatrix}$$

$$D^4u(x) = C^T\Delta^4\emptyset(x) = (c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3072 \end{bmatrix} = 3072\ c_4 \tag{2.9a}$$

by using the first root $x_r = 0.5$ of the polynomial $T^*_{m+1-\lceil\alpha\rceil}(x)$,we have that

$D^{\frac{7}{2}}u(x) = C^T\Delta^{\frac{7}{2}}\emptyset(x)=$

$$(c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 61.299 & -12.26 & 5.254 & -2.919 \end{bmatrix}\begin{bmatrix} 1 \\ 2x-1 \\ 8x^2-8x+1 \\ 32x^3-48x^2+18x-1 \\ 128x^4-256x^3+160x^2-32x+1 \end{bmatrix} = 9.341\ c_4 \tag{2.9b}$$

and,

$u^3(x) = (C^T\emptyset(x))^3$

$$= \left[(c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 1 \\ 2x-1 \\ 8x^2-8x+1 \\ 32x^3-48x^2+18x-1 \\ 128x^4-256x^3+160x^2-32x+1 \end{bmatrix}\right]^3 = (c_0 - c_2 + c_4)^3 \tag{2.9c}$$

by substitute (2.9a),(2.9b),(2.9c) in (2.7), we get

$$(2.10) \quad 3081.341\ c_4 + (c_0 - c_2 + c_4)^3 = 0.00195$$

from(2.8) ,we have that

$$c^T\emptyset(0) = (c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 1 \\ 2x-1 \\ 8x^2-8x+1 \\ 32x^3-48x^2+18x-1 \\ 128x^4-256x^3+160x^2-32x+1 \end{bmatrix}$$
$$= c_0 - c_1 + c_2 - c_3 + c_4 \tag{2.11}$$

and in derivative of zero are

$$c^T\emptyset^{(1)}(0) = (c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 0 \\ 2 \\ 16\,x-8 \\ 96\,x^2-96\,x+18 \\ 512\,x^3-768\,x^2+320\,x-32 \end{bmatrix}$$

$$= 2\,c_1 - 8\,c_2 + 18\,c_3 - 32\,c_4 \qquad (2.12)$$

$$c^T \emptyset^{(2)}(0) = (c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 0 \\ 0 \\ 16 \\ 192\,x - 96 \\ 1536\,x^2 - 1536\,x + 320 \end{bmatrix}$$

$$= 16\,c_2 - 96\,c_3 + 320\,c_4 \qquad (2.13)$$

$$c^T \emptyset^{3}(0) = (c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 192 \\ 3027\,x - 1536 \end{bmatrix}$$

$$= 192\,c_3 - 1536\,c_4 \qquad (2.14)$$

by(2.14)and(2.13),we get

$$c_3 = \frac{1536\,c_4 + 6}{192} = 8\,c_4 + \frac{1}{32} \quad , \quad c_2 = 28\,c_4 + \frac{3}{16} \qquad (2.15a)$$

from(2.12)and(2.11),we have

$$c_1 = \frac{32}{2} c_4 + \left[-18\left(\frac{1536\,c_4 + 6}{192\,(2)}\right)\right] + \left[-c_4 \frac{320}{2(16)} + \frac{8[96\,(1536\,c_4 + 6)]}{2 \cdot (16)(192)}\right]$$

$$=126\,c_4 + \frac{15}{32} \qquad (2.15b)$$

$$c_0 = -c_4 - 28\,c_4 - \frac{3}{16} + \left(8\,c_4 + \frac{1}{32}\right) + \left(126\,c_4 + \frac{15}{32}\right) = 105\,c_4 + \frac{5}{16}$$

(2.15c)

by substituting(2.15)(a),(b),(c)in (2.10), we get

$$3081.341\,c_4 + (78\,c_4 + 0.125)^3 - 0.00195 = 0 \qquad (2.16)$$

from(2.16), yield

$c_4 = 0$, and from (2.15)(a),(b),(c), we get $c_3 = \frac{1}{32}$, $c_2 = \frac{3}{16}$, $c_1 = \frac{15}{32}$, $c_0 = \frac{5}{16}$.

Hence, the approximate solution is

$$y(x) = \left(\frac{5}{16} \quad \frac{15}{32} \quad \frac{3}{16} \quad \frac{1}{32} \quad 0\right)\begin{bmatrix} 1 \\ 2x - 1 \\ 8x^2 - 8x + 1 \\ 32x^3 - 48x^2 + 18x - 1 \\ 128x^4 - 256x^3 + 160x^2 - 32x + 1 \end{bmatrix} = x^3$$

(2.17)

Theorem (2.1.7) :-

Let $\varphi(x)$ be shifted second kind Chebyshev vector defined in $\varphi(x) = [U_0^*(x), U_1^*(x), \cdots, U_n^*(x)]^T$ also suppose $\alpha > 0$ then

$D^\alpha \varphi(x) = \Delta^\alpha \varphi(x)$

where Δ^α is the $(m+1)(m+1)$ operational Matrix of fractional derivative of order α in the Caputo sense and defined as follows:

$$\Delta^{\alpha}=\begin{bmatrix} 0 & 0 & & 0 \\ . & . & & . \\ 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ \sum_{r=2}^{2} w_{2,0,r}^{(2)} & \sum_{r=2}^{2} w_{2,1,r}^{(2)} & \cdots & \sum_{r=2}^{2} w_{2,m,r}^{(2)} \\ \vdots & \vdots & & \vdots \\ \sum_{r=2}^{n+1} w_{n+1,0,r}^{(2)} & \sum_{r=2}^{n+1} w_{n+1,1,r}^{(2)} & \cdots & \sum_{r=2}^{n+1} w_{n+1,m,r}^{(2)} \\ \vdots & \vdots & & \vdots \\ \sum_{r=2}^{m} w_{m,0,r}^{(2)} & \sum_{r=2}^{m} w_{m,1,r}^{(2)} & \cdots & \sum_{r=2}^{m} w_{m,m,r}^{(2)} \end{bmatrix} \qquad (2.18)$$

and

$$w_{n+1,p,r}^{(2)} = \frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{n+p+2-(\ell+r)}\,(n+r)!(p+\ell)!\,(r-1)!\,2^{2(\ell+r)-2}\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(n+1-r)!\;\,2r!\,\Gamma(r-\alpha).(p+1-\ell)!\;\,2\ell!\,\Gamma(r-\alpha+\ell+1)} \qquad (2.19)$$

Proof:

Let $U_m^*(x)$ be Chebyshev polynomial of shifted second kind ,then by using (1.7) we can find that,

$D^{\alpha}U_n^*(x) = 0$, $n < \lceil\alpha\rceil$ and for $n \geq \lceil\alpha\rceil$ we have

$$D^{\alpha}U_n^*(x) = \sum_{r=0}^{n+1} r\,(-1)^{n+1-r}\,\frac{(n+r)!\;2^{2r-1}}{(n+1-r)!\;\;2r!}\,D^{\alpha}x^{r-1}$$

$$= \sum_{r=2}^{n+1} r\,(-1)^{n+1-r}\,\frac{(n+r)!\,2^{2r-1}(r-1)!}{(n+1-r)!\;\;2r!\,\lceil(r-\alpha)}\,x^{r-\alpha-1} \qquad (2.20)$$

Now, approximate $x^{r-\alpha-1}$ by $(m+1)$-terms of shifted second kind Chebyshev series, we have

$$x^{r-\alpha-1} = \sum_{p=0}^{m} d_{r-1,p}\;U_p^*(x) \qquad (2.21a)$$

where, $d_{r-1,p} = \frac{8}{\pi}\int_0^1 x^{r-\alpha-1}\sqrt{x-x^2}\,U_p^*(x)\,dx$ (2.21b)

$$= \frac{8}{\pi}\sum_{\ell=0}^{p+1}\frac{\ell(-1)^{p+1-\ell}\,(p+\ell)!\;\;2^{2\ell-1}}{(p+1-\ell)!\;\;2\ell!}\;x^{l-1}\int_0^1\sqrt{x-x^2}\,x^{r-\alpha-1}\;dx$$

$$= \frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{\ell(-1)^{p+1-\ell}\,(p+\ell)!\;\,2^{2\ell-1}\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(p+1-\ell)!\;\,2\ell!\,\Gamma(r-\alpha+\ell+1)}\,. \qquad (2.22)$$

Then,

$D^{\alpha}U_n^*(x)$

$$= \sum_{r=2}^{n+1}\sum_{p=0}^{m} r(-1)^{n+1-r}\,\frac{(n+r)!\,2^{2r-1}(r-1)!}{(n+1-r)!\;\,2r!\;\,\lceil(r-\alpha)}\,d_{r-1,p}\;U_n^*(x)$$

$$= \frac{4}{\sqrt{\pi}}\sum_{r=2}^{n+1}\sum_{p=0}^{m}\sum_{\ell=0}^{p+1}(-1)^{n+p+2-(l+r)}r.\,l\,\frac{2^{2(l+r)-2}(n+r)!\,(p+\ell)!(r-1)!\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(n+1-r)!(2r)!\Gamma(r-\alpha).(p+1-\ell)!\;\,(2\ell)!\,\Gamma(r-\alpha+\ell+1)}\,U_n^*(x)$$

where,

$$w^{(2)}_{n+1,p,r} = \frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{n+p+2-(\ell+r)}\,(n+r)!\,(p+\ell)!\;(r-1)!\;2^{2(\ell+r)-2}\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(n+1-r)!\;\;2r!\;\Gamma(r-\alpha).(p+1-\ell)!\;\;2\ell!\;\Gamma(r-\alpha+\ell+1)}$$

$$D^{\alpha}U_n^*(x) = \sum_{p=0}^{m}\left[\sum_{r=2}^{n+1} w^{(2)}_{n+1,p,r}\right] U_n^*(x) \qquad (2.23)$$

$$= \left[\sum_{r=2}^{n+1} w^{(2)}_{n+1,0,r}, \sum_{r=2}^{n+1} w^{(2)}_{n+1,1,r}, \cdots, \sum_{r=2}^{n+1} w^{(2)}_{n+1,m,r}\right]\varphi(x)$$

for, $n \geq \lceil\alpha\rceil$

and, $D^{\alpha}U_n^*(x) = [0,\cdots,0]\,\varphi(x) \qquad n < \lceil\alpha\rceil$.

2.2.2 Function Approximation of Shifted Chebyshev polynomial of Second Kind,[7]:

The function f(x) square integrable in [0,1] , may be expressed in the term of shifted second kind Chebyshev polynomial as

$$u(x) = \sum_{i=0}^{\infty} c_i\, U_i^*(x)$$

where the coefficients c_i are given by in(1.37)

In practice , only the first (m+1)-terms shifted second kind Chebyshev polynomial are considered**,** Then we have

$$u_m(x) = \sum_{i=0}^{m} c_i\, U_i^*(x) = C^T\varphi(x)$$

Example(2.1.7):

Consider the following multi-fractional order nonlinear differential equation

$$D^3u(x) + D^{\frac{5}{2}}u(x) + u^2(x) = x^4 \qquad (2.24)$$

subject to initial conditions,

$$u(0) = u^{(1)}(0) = 0\;, u^{(2)}(0) = 2 \qquad (2.25)$$

with exact solution $y(x) = x^2$. [44]

To solve the above problem with m=3. From(2.18), we have that

$$\Delta^{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{r=2}^{3} W^{(2)}_{3,r,0} & \sum_{r=2}^{3} W^{(2)}_{3,r,1} & \sum_{r=2}^{3} W^{(2)}_{3,r,2} & \sum_{r=2}^{3} W^{(2)}_{3,r,3} \end{bmatrix}$$

$$\Delta^{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.472\times10^{5} & 2.895\times10^{6} & 7.757\times10^{7} & 3.172\times10^{9} \end{bmatrix}$$

$$D^3u(x) = C^T\Delta^3\varphi(x) = (c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 384 \end{bmatrix} = 384\,c_3 \qquad (2.26a)$$

$$u^2(x) = (C^T\varphi(x))^2 = \left[(c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 1 \\ 4x-2 \\ 16x^2-16x+3 \\ 64x^3-96x^2+40x-4 \end{bmatrix}\right]^2$$

by using the first $x_r = \frac{1}{2}$ of root $U^*_{m+1-\lceil\alpha\rceil}(x)$

$D^{\frac{5}{2}}u(x) = C^T D^{\frac{5}{2}}\varphi(x) =$

$$(c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.472\times 10^5 & 2.895\times 10^6 & 7.757\times 10^7 & 3.172\times 10^9 \end{bmatrix}\begin{bmatrix} 1 \\ 4x-2 \\ 16x^2-16x+3 \\ 64x^3-96x^2+40x-4 \end{bmatrix}$$

$$= 247200\, c_3 - 77570000\, c_3 = -77322800\, c_3 \tag{2.26b}$$

$$u^2(x) = (C^T\varphi(x))^2 = \left[(c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 1 \\ 4x-2 \\ 16x^2-16x+3 \\ 64x^3-96x^2+40x-4 \end{bmatrix}\right]^2$$

$$= (c_0 - c_2)^2 \tag{2.26c}$$

from(2.25),we have

$$c_0 - 2c_1 + 3c_2 - 4c_3 = 0 \tag{2.27}$$

$$4c_1 - 16c_2 + 40c_3 = 0 \tag{2.28}$$

$$32c_2 - 192c_3 = 2 \tag{2.29}$$

from(2.29)and(2.28),we get

$$c_2 = \frac{192c_3+2}{32} = 6c_3 + \frac{1}{16} \quad \text{thus} \quad c_2 = 6c_3 + \frac{1}{16} \tag{2.30a}$$

$$4c_1 - 16\left(6c_3 + \frac{1}{16}\right) + 40c_3 = 0$$

$$4c_1 - 96c_3 - 1 + 40c_3 = 0 \ \text{ yelid } \ c_1 = 14c_3 + \frac{1}{4} \tag{2.30b}$$

from(2.27)and(2.26),we obtain

$$c_0 - 2\left(14c_3 + \frac{1}{4}\right) + 3\left(6c_3 + \frac{1}{16}\right) - 4c_3 = 0$$

$$c_0 - \frac{1}{2} - 28c_3 + 18c_3 + \frac{3}{16} - 4c_3 = 0 \ \text{ thus } \ c_0 = 14c_3 + \frac{5}{16} \tag{2.30c}$$

By substituting(2.30)(a),(b)and(c) in (2.24), we get

$$-77322416\, c_3 + \left(8c_3 + \frac{1}{4}\right)^2 - 0.0625 = 0 \tag{2.31}$$

by(2.31), we get

$c_3 = 0$,and from(2.30)(a), (b), (c), and(d), we get ,

$c_2 = \frac{1}{16}$, $c_1 = \frac{1}{4}$, $c_0 = \frac{5}{16}$.

Hence, the approximate solution is

$$y(x) = \frac{5}{16} + \frac{1}{4}(4x-2) + \frac{1}{16}(16x^2-16x+3) + 0$$

$$= \frac{8}{16} - \frac{1}{2} + x^2 = x^2$$

2.3 The Operational Matrix of Fractional Derivative for Chebyshev's Wavelets:

New proposed operational matrix formulation of fractional order derivative α>0 for shifted first and second kinds Chebyshev wavelets denoted by $D^{\alpha}\Psi^{1}_{nm}(t)$, $D^{\alpha}\Psi^{2}_{nm}(t)$.

2.3.1 The Operational Matrix of Fractional Derivative for Chebyshev Wavelets of First Kind:

The shifted Chebyshev wavelets of first kind $\Psi^{1}_{nm}(t) = \Psi^{1}(k, \tilde{n}, m, t)$ have four arguments; $k \in \mathbb{N}$, n=1,2,…, 2^{k-1} and $\tilde{n} = 2n - 1$; moreover, m is the order of the Chebyshev polynomials of first kind and t is the normalized time ,where it is generalized of [16]

$$\Psi^{1}_{\tilde{n}m}(t) = \begin{cases} 2^{k/2} T^{*}_{m}(2^{k}t - \tilde{n}) & \frac{\tilde{n}-1}{2^{k}} \leq t \leq \frac{\tilde{n}+1}{2^{k}} \\ 0 & o.w \end{cases}$$

(2.32a)

$$\text{Where,} \quad T^{*}_{m} = \begin{cases} \frac{1}{\sqrt{\pi}} T_{m} & m = 0 \\ \sqrt{\frac{2}{\pi}} T_{m} & m > 0 \end{cases}$$

(2.32b)

m = 0,1,…….M , n = 1,2,……2^{k-1} .

(2.32c)

The weight is $\omega_{\tilde{n}}(t) = \omega^{*}_{1,n}(2^{k}t - \tilde{n})$.

Where $\omega^{*}_{1,n}(2^{k}t - \tilde{n}) = \frac{1}{\sqrt{(2k\,t - \tilde{n}) - (2k\,t - \tilde{n})^{2}}}$.

2.3.2 Function Approximation of Shifted Chebyshev Wavelet of First Kind:

A function f(t) defined over [0,1], may be expanded as follows:

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\,\Psi^{1}_{\tilde{n}m}(t) , \qquad (2.33a)$$

where, $c_{\tilde{n}m} = (f(t) , \Psi^{1}_{\tilde{n}m}(t))_{w} = \int_{0}^{1} \tilde{w}(t)\, f(t)\, \Psi^{1}_{\tilde{n}m}(t)\, dt$

and,

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{\tilde{n}m}\, \Psi^{1}_{nm}(t) = c^{T}\, \Psi^{1}_{\tilde{n}m}(t) \qquad (2.33b)$$

where, $C =[c_{10} , c_{11} , \dots . c_{2^{k-1},M} , \dots . c_{2^{k-1},1} , \dots . c_{2^{k-1},M}]^{T}$

Thus,

$$\Psi^{1}_{\tilde{n}m}(t) =[\Psi^{1}_{1,0} , \Psi^{1}_{1,1} , \dots , \Psi^{1}_{1,M}, \dots , \Psi^{1}_{2^{k-1},M}, \dots , \Psi^{1}_{2^{k-1},1} , \dots , \Psi^{1}_{2^{k-1},M}]^{T}$$

Theorem(2.3.8) :

Let $\Psi^{1}_{\tilde{n}m}(t)$ be shifted Chebyshev wavelets vector of first kind and also suppose $\alpha > 0$, then

$D^{\alpha}\Psi^{1}_{\tilde{n}m}(t)(t) = D^{\alpha}\left(\frac{c_m . 2^{k/2}}{\sqrt{\pi}} T^{*}_{m}((2^k t - \tilde{n})\right) = \Delta^{(\alpha)}\ \Psi^{1}_{nm}(t)$, such that $c_m = \begin{cases} 1 & m = 0 \\ \sqrt{2} & m > 0 \end{cases}$, where Δ^{α} is the (m+1) (m+1) operational matrix derivative of order (α) in the Caputo sense and is defined as follows ,

$$\Delta^{\alpha} = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ w^{\sim}_{0,0,i} & w^{\sim}_{0,1,i} & \cdots & w^{\sim}_{0,m,i} \\ \vdots & \vdots & & \vdots \\ \sum_{i=0}^{m-\lceil\alpha\rceil} w^{\sim}_{m-\lceil\alpha\rceil,0,i} & \sum_{i=0}^{m-\lceil\alpha\rceil} w^{\sim}_{m-\lceil\alpha\rceil,1,i} & \cdots & \sum_{i=0}^{m-\lceil\alpha\rceil} w^{\sim}_{m-\lceil\alpha\rceil,m,i} \\ \vdots & \vdots & & \vdots \\ \sum_{i=0}^{m} w^{\sim}_{m,0,i} & \sum_{i=0}^{m} w^{\sim}_{m,1,i} & \cdots & \sum_{i=0}^{m} w^{\sim}_{m,m,i} \end{bmatrix}$$

(2.34)

and $w^{\sim}_{m-\lceil\alpha\rceil,j,i}$ is given by

$$w^{\sim}_{m-\lceil\alpha\rceil,j,i} = \frac{\sigma_j}{\sqrt{\pi}} \sum_{k=0}^{j} (-1)^{k+i}\, 2^{2(j+m)-2(k+i)} \frac{m(2m-i-1)!(m-i)!j(2j-k-1)!\,\Gamma(m-i-\alpha+j-k+\frac{1}{2})}{i!(2m-2i)!\,\Gamma(m-i-\alpha+1)k!(2j-2k)!\,\Gamma(m-i-\alpha+k+1)}$$

(2.35)

where, $\sigma_j = \begin{cases} 1 & j = 0 \\ 2 & j \neq 0 \end{cases}$

Proof:

Let $T^{*}_{m}\ (2^k t - \tilde{n})$ be shifted Chebyshev wavelets of first kind. Then by substituting (1.6)and(1.7)in remark(1.4.4), we get

$D^{\alpha}T^{*}_{m}\ (2^k t - \tilde{n}) = 0 \qquad m < \lceil\alpha\rceil$, and for $m \geq \lceil\alpha\rceil$ we get

$$D^{\alpha}\, T^{*}_{m}\ (2^k t - \tilde{n}) = \sum_{i=0}^{m} (-1)^{i}\, 2^{2m-2i} \frac{m(2m-i-1)!}{i!(2m-2i)!} D^{\alpha}\,(2^k t - \tilde{n})^{m-i}$$

from(1.8), if we replacing $x = 2^k t$ and $a = \tilde{n}$,

we get

$$= \sum_{i=0}^{m-\lceil\alpha\rceil} (-1)^{i}\, 2^{2m-2i} \frac{m(2m-i-1)!(m-i+1)!}{i!(2n-2i)!\,\Gamma(m-i-\alpha+1)} (2^k t - \tilde{n})^{m-i-\alpha} \qquad (2.36)$$

now, approximate $(2^k t - \tilde{n})^{m-i-\alpha}$ by (m+1) terms shifted chebyshev wavelets of first kind, we have

$$(2^k t - \tilde{n})^{m-i-\alpha} = \sum_{j=0}^{m} d_{m-i,j}\, T^{*}_{j}(2^k t - \tilde{n}) \qquad (2.37a)$$

where, $d_{m-i,j} = \frac{\sigma_j}{\pi} \int_0^1 (2^k t - \tilde{n})^{m-i-\alpha} \frac{T^{*}_{j}\ (2^k t - \tilde{n})}{\sqrt{(2^k t - \tilde{n}) - (2^k t - \tilde{n})^2}} dt$

$$= \frac{\sigma_j}{\pi} \sum_{k=0}^{j} (-1)^{k}\, 2^{2j-2k} \frac{j(2j-k-1)!}{k!(2j-2k)!} \int_0^1 \frac{(2^k t - \tilde{n})^{m-i-\alpha+j-k}}{\sqrt{(2^k t - \tilde{n}) - (2^k t - \tilde{n})^2}} dt$$

where, $\sigma_j = \begin{cases} 1 & j = 0 \\ 2 & j \neq 0 \end{cases}$.

Then,

$$d_{m-i,j} = \frac{\sigma\, j}{\sqrt{\pi}} \sum_{k=0}^{j} (-1)^k\, 2^{2j-2k}\; \frac{j(2j-k-1)!\;\Gamma(m-i-\alpha+j-k+\frac{1}{2})}{k!(2j-2k)!\;\Gamma(m-i-\alpha+k+1)} \qquad (2.37b)$$

Therefore ,

$D^{\alpha}T_m^{*}\,(2^k t - \tilde{n})$

$$=\sum_{i=0}^{m-\lceil\alpha\rceil} \sum_{j=0}^{m} (-1)^i\, 2^{2m-2i} \frac{m(2m-i-1)!(m-i)!}{i!(2m-2i)!\Gamma(m-i-\alpha+1)}\; d_{m-i,j} T_m^{*}\,(2^k\, t - \tilde{n})$$

$$=\sum_{j=0}^{m} \left[\sum_{i=0}^{m-\lceil\alpha\rceil} w_{m,j,i}\right]\; T_m^{*}(2^k t\text{-}\tilde{n}). \qquad (2.38)$$

$D^{\alpha}\Psi_{\tilde{n}m}^{1}(t)=$

$$\left[\sum_{i=0}^{m-\lceil\alpha\rceil} w_{m-\lceil\alpha\rceil,0,i}^{\sim},\; \sum_{i=0}^{m-\lceil\alpha\rceil} w_{m-\lceil\alpha\rceil,1,i}^{\sim}, \ldots, \sum_{i=0}^{m-\lceil\alpha\rceil} w_{m-\lceil\alpha\rceil m,i}^{\sim}\right]\Psi_{\tilde{n}m}^{1}(t)$$

for, $m \geq \lceil\alpha\rceil$

and

$D^{\alpha}\Psi_{\tilde{n}m}^{1}(t) = [0,\ldots.0]\; \Psi_{\tilde{n}m}^{1}(t)$ for $m < \lceil\alpha\rceil$. (2.39)

Example(2.3.8):

Consider the following multi-fractional order nonlinear differential equation

$$D^3u(x) + D^{\frac{3}{2}}u(x) + D^{\frac{1}{2}}u(x) + u^2(x) = 2.257\, x^{\frac{1}{2}} + 1.505\, x^{\frac{3}{2}} + x^4 \qquad (2.40)$$

with initial condition,

$u(0) = 0$, $u^{(1)}(0) = 0$, $u^{(2)}(0) = 2$ (2.41)

and, $m = 3$, $k = 1$, with exact solution $u(x) = x^2$. by use(2.32)(a),(c),the wavelets polynomial with $m = 0,1,2,3$ and n=1

$$\psi_{1,0}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} T_0(2x-1) = \sqrt{\frac{2}{\pi}}$$

$$\psi_{1,1}(x) = \frac{2}{\sqrt{\pi}} T_1(2x-1) = \frac{2}{\sqrt{\pi}}(2x-1)$$

$$\psi_{1,2}(x) = \frac{2}{\sqrt{\pi}} T_2(2x-1) = \frac{2}{\sqrt{\pi}}(8x^2-8x+1)$$

$$\psi_{1,3}(x) = \frac{2}{\sqrt{\pi}} T_3(2x-1) = \frac{2}{\sqrt{\pi}}(32x^3-48x^2+18$$

by(2.34), we have that

$$\Delta^{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sum_{i=0}^{1} w_{1,0,i}^{\sim} & \sum_{i=0}^{1} w_{1,1,i}^{\sim} & \sum_{i=0}^{1} w_{1,2,i}^{\sim} & \sum_{i=0}^{1} w_{1,3,i}^{\sim} \\ \sum_{i=0}^{2} w_{1,0,i}^{\sim} & \sum_{i=0}^{2} w_{1,1,i}^{\sim} & \sum_{i=0}^{2} w_{1,2,i}^{\sim} & \sum_{i=0}^{2} w_{1,3,i}^{\sim} \\ \sum_{i=0}^{3} w_{3,0,i}^{\sim} & \sum_{i=0}^{3} w_{3,1,i}^{\sim} & \sum_{i=0}^{3} w_{3,2,i}^{\sim} & \sum_{i=0}^{3} w_{3,3,i}^{\sim} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3.831 & -0.766 & 0.328 \\ 0 & -10.727 & 6.349 & -2.165 \end{bmatrix}$$

and,

$$\Delta^{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{i=0}^{2} w_{1,0,i}^{\sim} & \sum_{i=0}^{2} w_{1,1,i}^{\sim} & \sum_{i=0}^{2} w_{1,2,i}^{\sim} & \sum_{i=0}^{2} w_{1,3,i}^{\sim} \\ \sum_{i=0}^{3} w_{2,0,i}^{\sim} & \sum_{i=0}^{3} w_{2,1,i}^{\sim} & \sum_{i=0}^{3} w_{2,2,i}^{\sim} & \sum_{i=0}^{3} w_{2,3,i}^{\sim} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.958 & -0.192 & 0.082 \\ 0 & -0.766 & 1.204 & -0.377 \\ 0 & -5.099 & -2.794 & 1.172 \end{bmatrix}$$

$$D^3u(x) = C^T\Delta^3\Psi_{nm}^1(t) = (c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 271.5290 \end{bmatrix}$$

$$= 271.5290\, c_3 \qquad (2.42a)$$

by using the first root $x_r = 0.8$ of $T_{m+1-2}^{*}(x)$

$$D^{\frac{3}{2}}u(x) = C^T\Psi_{nm}^1(t) =$$

$$(c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3.831 & -0.766 & 0.328 \\ 0 & -10.727 & 6.349 & -2.165 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\frac{2}{\pi}} \\ \frac{2}{\sqrt{\pi}}(2x-1) \\ \frac{2}{\sqrt{\pi}}(8x^2-8x+1) \\ \frac{2}{\sqrt{\pi}}(32x^3-48x^2+18x-1) \end{bmatrix}$$

$$= 2.4892\, c_2 - 6.9818\, c_3 \qquad (2.42b)$$

$$D^{\frac{1}{2}}u(x) = C^T D^{\frac{1}{2}}\Psi_{nm}^1(t) =$$

$$
(c_0 \quad c_1 \quad c_2 \quad c_3)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.958 & -0.192 & 0.082 \\ 0 & -0.766 & 1.204 & -0.377 \\ 0 & -5.099 & -2.794 & 1.172 \end{bmatrix}
$$

$$
\cdot \begin{bmatrix} \sqrt{\frac{2}{\pi}} \\ \frac{2}{\sqrt{\pi}}(2x-1) \\ \frac{2}{\sqrt{\pi}}(8x^2-8x+1) \\ \frac{2}{\sqrt{\pi}}(32x^3-48x^2+18x-1) \end{bmatrix}
$$

$$
= 0.6225\, c_1 - 0.5008\, c_2 - 3.8072\, c_3 \qquad (2.42c)
$$

$$
u^2(x) = \left(c^T \Psi_{nm}^{1}(t)\right)^2
$$

$$
= (0.6770\, c_1 - 1.0561\, c_3 - 0.3159\, c_2 + 0.797\, c_0)^2
$$

by substituting (2.42)(a),(b),(c)and(d)in (2.40), we get

$$
0.6225\, c_1 + 1.9884\, c_2 + 260.74\, c_3 + (0.6770\, c_1 - 1.0561\, c_3 - 0.3159\, c_2 + 0.797\, c_0)^2 = 3.505 \qquad (2.43)
$$

and, from(2.41), we have

$$
\sqrt{\frac{2}{\pi}}\, c_0 - \frac{2}{\sqrt{\pi}}\, c_1 + \frac{2}{\sqrt{\pi}}\, c_2 - \frac{2}{\sqrt{\pi}}\, c_3 = 0 \qquad (2.44)
$$

$$
\frac{4}{\sqrt{\pi}}\, c_1 - \frac{16}{\sqrt{\pi}}\, c_2 + \frac{36}{\sqrt{\pi}}\, c_3 = 0 \qquad (2.45)
$$

$$
\frac{32}{\sqrt{\pi}}\, c_2 - \frac{192}{\sqrt{\pi}}\, c_3 = 2 \qquad (2.46)
$$

$c_0 = 0.5888$, $c_1 = 0.5694$, $c_2 = 0.1613$, $c_3 = 0.0084$

$$
y(x) = c_0\psi_{1,0}(x) + c_1\psi_{1,1}(x) + c_2\psi_{1,2}(x) + c_3\psi_{1,3}(x)
$$

$$
= 0.5888\left(\sqrt{\frac{2}{\pi}}\right) + 0.5694\left[\frac{2}{\sqrt{\pi}}(2x-1)\right] + 0.1613\left[\frac{2}{\sqrt{\pi}}(8x^2 - 8x+1)\right] + 0.0084\left[\frac{2}{\sqrt{\pi}}(32x^3-48x^2+18x-1)\right]
$$

$$
= 0.47 + 0.642\,(2x-1) + 0.182\,(8x^2-8x+1) + 0.0094\,(32x^3-48x^2+18x-1)
$$

$$
= 0.0006 - 0.0028\,x + 1.0048\,x^2 + 0.3008\,x^3 \qquad (2.47)
$$

Table(2.1)

x	Exact solution	Approximate solution
0.1	0.01	0.011
0.2	0.04	0.043
0.3	0.09	0.098
0.4	0.16	0.179
0.5	0.25	0.288
0.6	0.36	0.426

0.7	0.49	0.594
0.8	0.64	0.795
0.9	0.81	1.031

2.3.3 The Operational Matrix of Fractional Derivative for Chebyshev Wavelets Of Second Kind,[29]:

The second kind Chebyshev wavelets $\Psi^2_{nm}(t) = \Psi^2$ (k,n,m,t) have four arguments k,n can assume any positive integer , m is the order of second kind Chebyshev polynomials ,and t is the normalized time. They are defined on the interval [0,1]

$$\Psi^2_{nm}(t)=\begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m\left(2^k t - n\right) & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}] \\ 0 & o.w \end{cases} \quad (2.48a)$$

m=0,1,……M,n=0,1,…… 2^k-1 (2.48b)

The weight function is $\omega_n(t) = \omega^*_{2,n}(2^k t - n)$

where, $\omega^*_{2,n}(2^k t - n) = \sqrt{(2^k t - n) - (2^k t - n)^2}$

2.3.4 Function Approximation of Shifted Chebyshev Wavelet of Second Kind,[29] :

A function (t) defined over[0, 1] may be expanded in terms for Chebyshev wavelets of second kind as

$$f(t)=\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \Psi^2_{nm}(t) \quad (2.49a)$$

If the infinite series is truncated, then function approximate for f(t) can be expressed as

$$f(t)=\sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm} \Psi_{nm}(t)=c^T\Psi^2_{nm}(t) \quad (2.49b)$$

where , $c_{nm} = (f(t), \Psi^2_{nm}(t)) = \int_0^1 \omega_n(t). f(t). \Psi^2_{nm}(t)\, dt$

since, C and $\Psi(t)$ are $2^k (M + 1) \times 1$ matrices defined by

$C = [\, c_{00}, c_{01}, \ldots\ldots c_{2^k-1,M}, \ldots\ldots c_{2^k-1,0}, c_{2^k-1,1}, \ldots\ldots c_{2^k-1,M} \,]^T$

$\Psi^2_{nm}(t) = [\Psi^2_{0,0}, \Psi^2_{0,1}, \ldots, \Psi^2_{0,M}, . -, \Psi^2_{2^k-1,M}, \ldots \Psi^2_{2^k-1,0}, \Psi^2_{2^k-1,1}, \ldots, \Psi^2_{2^k-1,M}\,]$

Theorem(2.3.9):

Let $\Psi^2_{nm}(t)$ be second kind Chebyshev wavelets and suppose $\alpha > 0$,then $D^{\alpha}\Psi^2_{nm}(t) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} D^{\alpha}(U^*_m(2^k t - n) = \Delta^{\alpha}\Psi^2_{nm}(t)$, where Δ^{α} is (m+1) (m+1) operational matrix derivative of order α in the Caputo sense and defined by,

$$\Delta^{\alpha}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ . & & . & . \\ 0 & 0 & \cdots & 0 \\ \sum_{r=2}^{2} w_{2,0,r}^{\sim} & \sum_{r=2}^{2} w_{2,1,r}^{\sim} & \cdots & \sum_{r=2}^{2} w_{2,m,r}^{\sim} \\ \vdots & \vdots & & \vdots \\ \sum_{r=2}^{n+1} w_{n+1,0,r}^{\sim} & \sum_{r=2}^{n+1} w_{n+1,1,r}^{\sim} & \cdots & \sum_{r=2}^{n} w_{n+1,m,r}^{\sim} \\ \vdots & \vdots & & \vdots \\ \sum_{r=2}^{m} w_{m,0,r}^{\sim} & \sum_{r=2}^{m} w_{m,1,r}^{\sim} & \cdots & \sum_{r=2}^{m} w_{m,m,r}^{\sim} \end{bmatrix}$$

(2.50)

and, $w_{n+1,p,r}^{\sim}=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2(\ell+r)-2} r\ell(-1)^{n+p+2-(\ell+r)}(r-1)!(p+\ell)!\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(n+1-r)!\ 2r!\ \Gamma(r-\alpha)(p+1-\ell)!\ 2\ell!\,\Gamma(r-\alpha+\ell+1)}$

Proof :

Let $U_m^*(2^k\,t-n)$ be shifted Chebyshev wavelets of second kind. Then by substituting (1.6)and(1.7)in remark(1.36c), we get

$$U_m^*(2^k\,t-n)=\sum_{r=0}^{m+1} r\,(-1)^{m+1-r}\frac{(m+r)!\,2^{2r-1}}{(m+1-r)!\ 2r!}(2^k\,t-n)^{r-1}$$

also , we have that $D^{\alpha}U_n^*(2^k\,t-n)=0 \quad n<\lceil\alpha\rceil$

and for $n\geq\lceil\alpha\rceil$, we have

$$D^{\alpha}U_n^*(2^k\,t-n)=\sum_{r=0}^{n+1} r\,(-1)^{n+1-r}\frac{(n+r)!\,2^{2r-1}}{(n+1-r)!\ 2r!}D^{\alpha}(2^k\,t-n)^{r-1}$$

$$=\sum_{r=2}^{n+1} r\,(-1)^{n+1-r}\frac{(n+r)!\,2^{2r-1}(r-1)!}{(n+1-r)!\ 2r!\ \Gamma(r-\alpha)}(2^k\,t-n)^{r-\alpha-1} \qquad (2.51)$$

now, approximate $(2^k\,t-n)^{r-\alpha-1}$ by (m+1) – term for shifted chebyshev wavelets for second kind, we have

$$(2^k\,t-n)^{r-\alpha-1}=\sum_{p=0}^{m} d_{r-1,p}\,U_p^*(2^k\,t-n) \qquad (2.52a)$$

$$d_{r-1,p}=\frac{8}{\pi}\int_0^1(2^k\,t-n)^{r-\alpha-1}\;U_p^*\;(2^k\,t-n)\sqrt{(2^k\,t-n)-(2^k\,t-n)^2}\;dt$$

$$=\frac{4}{\pi}\sum_{\ell=0}^{p+1}\frac{2^{2\ell-1}(-1)^{p+1-\ell}\,\ell\,(p+\ell)!}{(p+1-\ell)!\,2\ell!\,\Gamma(r+\ell-\alpha)}\int_0^1\sqrt{(2^k\,t-n)-(2^k\,t-n)^2}.\,(2^k\,t-n)^{r-\alpha+l-2}\,dt$$

$$=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2\ell-1}\,\ell\,(-1)^{p+1-\ell}\,(p+\ell)!\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(p+1-\ell)!\,2\ell!\,\Gamma(r+\ell-\alpha+1)} \qquad (2.52b)$$

Therefore,

$$D^{\alpha}U_n^*(2^k\,t-n)=\sum_{r=2}^{n+1}\sum_{p=0}^{m} r\,(-1)^{n+1-r}\frac{(m+r)!\,2^{2r-1}(r-1)!}{(m+1-r)!\ 2r!\ \Gamma(r-\alpha)}d_{r-1,p}.\,U_n^*\;(2^k\,t-n)$$

$$=\sum_{p=0}^{n}\left[\sum_{r=2}^{n+1} w_{n+1,p,r}^{\sim}\right]U_n^*(2^k\,t-n) \qquad (2.53)$$

$D^{\alpha}\Psi_{nm}^2(t)=\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\sum_{p=0}^{m}\left[\sum_{r=2}^{n+1} w_{n+1,p,r}^{\sim}\right]U_n^*(2^k\,t-n)$, thus

$D^{\alpha}\Psi_{nm}^2(t)=[\,\sum_{r=2}^{n+1} w_{n+1,0,r}^{\sim},\sum_{r=2}^{n+1} w_{n+1,1,r}^{\sim},\ldots,\sum_{r=2}^{n+1} w_{n+1,m,r}^{\sim}\,].\,\Psi_{nm}^2(t)$

$$\text{And, } D^{\alpha}\Psi_{nm}^2(t)=\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}D^{\alpha}U_n^*(2^k\,t-n)=[\,0,0,\ldots,0\,]\,\Psi_{nm}^2(t),\ n\leq\lceil\alpha\rceil \qquad (2.54)$$

Example(2.3.9):

Consider the following multi-fractional order nonlinear differential equation

$$D^4u(x)+D^{\frac{7}{2}}u(x)+D^{\frac{5}{2}}u(x)+u^3(x)=6.772\,x^{\frac{1}{2}}+x^9 \tag{2.55}$$

with initial condition

$$u(0)=u^{(1)}(0)=u^{(2)}(0)=0\quad,\quad u^{(3)}(0)=6 \tag{2.56}$$

with exact solution $y(x)=x^3$, $m=4$, $k=0$

by use(2.48a)and (2.48b),we get $m=0,1,2,3,4$ and $n=0$

$$\psi_{0,0}(x)=2\sqrt{\frac{2}{\pi}}\cdot U_0^*(x)=2\sqrt{\frac{2}{\pi}}$$

$$\psi_{0,1}(x)=2\sqrt{\frac{2}{\pi}}\cdot U_1^*(x)=\sqrt{\frac{2}{\pi}}(8\,x-4)$$

$$\psi_{0,2}(x)=2\sqrt{\frac{2}{\pi}}\cdot U_2^*(x)=\sqrt{\frac{2}{\pi}}(32\,x^2-32\,x+6)$$

$$\psi_{0,3}(x)=2\sqrt{\frac{2}{\pi}}\cdot U_3^*(x)=2\sqrt{\frac{2}{\pi}}(64\,x^3-96\,x^2+40\,x-4)$$

$$\psi_{0,4}(x)=2\sqrt{\frac{2}{\pi}}\cdot U_4^*(x)=2\sqrt{\frac{2}{\pi}}(256\,x^4-512\,x^3+336\,x^2-80\,x+5)$$

$$\Delta^{\frac{7}{2}}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\\sum_{r=2}^{4}w_{4,0,r}^{\sim}&\sum_{r=2}^{4}w_{4,1,r}^{\sim}&\sum_{r=2}^{4}w_{4,2,r}^{\sim}&\sum_{r=2}^{4}w_{4,3,r}^{\sim}&\sum_{r=2}^{4}w_{4,4,r}^{\sim}\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\7.118\times10^{7}&8.338\times10^{8}&2.234\times10^{10}&9.135\times10^{11}&5.077\times10^{13}\end{bmatrix}$$

$$\Delta^{\frac{5}{2}}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\\sum_{r=2}^{3}w_{3,0,r}^{\sim}&\sum_{r=2}^{3}w_{3,1,r}^{\sim}&\sum_{r=2}^{3}w_{3,2,r}^{\sim}&\sum_{r=2}^{3}w_{3,3,r}^{\sim}&\sum_{r=2}^{3}w_{3,4,r}^{\sim}\\\sum_{r=2}^{4}w_{4,0,r}^{\sim}&\sum_{r=2}^{4}w_{3,1,r}^{\sim}&\sum_{r=2}^{4}w_{3,2,r}^{\sim}&\sum_{r=2}^{4}w_{3,3,r}^{\sim}&\sum_{r=2}^{4}w_{3,4,r}^{\sim}\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\-1.977\times10^{6}&-2.316\times10^{7}&-6.206\times10^{8}&-2.538\times10^{10}&-1.41\times10^{12}\\2.514\times10^{7}&3.565\times10^{8}&1.035\times10^{10}&4.468\times10^{11}&2.581\times10^{13}\end{bmatrix}$$

$D^4u(x) = C^T D^4 \Psi^2_{nm}(t) =$

$$(c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4) 2\sqrt{\frac{2}{\pi}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6144 \end{bmatrix} = 9804.4054\ c_4$$

(2.57a)

by using the first root $x_r = \frac{1}{2}\ U^*_{m+1-2}(x)$

$$c^T \cdot D^{\frac{7}{2}} \cdot \psi(x) = -8.8147 \times 10^{10} \cdot \sqrt{2}\, c_4 + 5.7287 \times 10^{13} \cdot \sqrt{2}\, c_4$$
$$= 8.0891 \times 10^{13}\ c_4$$

(2.57b)

$$c^T \cdot D^{\frac{5}{2}} \cdot \psi(x) = -2.2463 \times 10^{12} c_3 + 4.1125 \times 10^{13} c_4$$

by substituting(2.57a),(b)and(c)in(2.55),we get

$$-2.2463 \times 10^{12} c_3 + 1.2201 \times 10^{14} c_4 + (1.5956\ c_0 - 1.5956\ c_4 - 5.5851\ c_2)^3 = 4.79 \quad (2.58)$$

also, from(2.55),we get

$$2\sqrt{\frac{2}{\pi}}\ c_0 - 4\sqrt{\frac{2}{\pi}}\ c_1 + 6\sqrt{\frac{2}{\pi}}\ c_2 - 8\sqrt{\frac{2}{\pi}}\ c_3 + 10\sqrt{\frac{2}{\pi}}\ c_4 = 0 \quad (2.59)$$

$$8\sqrt{\frac{2}{\pi}}\ c_1 - 32\sqrt{\frac{2}{\pi}}\ c_2 + 80\sqrt{\frac{2}{\pi}}\ c_3 - 160\sqrt{\frac{2}{\pi}}\ c_4 = 0 \quad (2.60)$$

$$64\sqrt{\frac{2}{\pi}}\ c_2 - 384\sqrt{\frac{2}{\pi}}\ c_3 + 1344\sqrt{\frac{2}{\pi}}\ c_4 = 0 \quad (2.61)$$

$$768\sqrt{\frac{2}{\pi}}\ c_3 - 12288\sqrt{\frac{2}{\pi}}\ c_4 = 6 \quad (2.62)$$

from(2.62)and(2.61),we have $768\sqrt{\frac{2}{\pi}}\ c_3 = 12288\sqrt{\frac{2}{\pi}}\ c_4 + 6$

$$c_3 = 16\ c_4 + \frac{\sqrt{\pi}}{128\sqrt{2}} \quad (2.63a)$$

$$64\sqrt{\frac{2}{\pi}}\ c_2 - 384\sqrt{\frac{2}{\pi}}\ c_2 \left(16\ c_4 + \frac{\sqrt{\pi}}{128\sqrt{2}}\right) + 1344\sqrt{\frac{2}{\pi}}\ c_4 = 0$$

$$64\sqrt{\frac{2}{\pi}}\ c_2 - 6144000\sqrt{\frac{2}{\pi}}\ c_4 - 3 + 1344\sqrt{\frac{2}{\pi}}\ c_4 = 0$$

$$64\sqrt{\frac{2}{\pi}}\ c_2 = 6143 \times 10^6 \sqrt{\frac{2}{\pi}}\ c_4 + 3$$

$$c_2 = 95980\ c_4 + \frac{3\sqrt{\pi}}{64\sqrt{2}} \quad (2.63b)$$

and ,from (2.59),(2.60),we get

$$8\sqrt{\frac{2}{\pi}}\ c_1 - 32\sqrt{\frac{2}{\pi}}\left(95980\ c_4 + \frac{3\sqrt{\pi}}{64\sqrt{2}}\right) + 80\sqrt{\frac{2}{\pi}}\left(16\ c_4 + \frac{\sqrt{\pi}}{128\sqrt{2}}\right) - 160\sqrt{\frac{2}{\pi}}\ c_4 = 0$$

$$8\sqrt{\frac{2}{\pi}}\ c_1 - 3071000\sqrt{\frac{2}{\pi}}c_4 - \frac{96}{64} + 1280\sqrt{\frac{2}{\pi}}\ c_4 + \frac{80}{128} - 160\sqrt{\frac{2}{\pi}}\ c_4 = 0$$

$$8\sqrt{\frac{2}{\pi}}\ c_1 = 3070000\sqrt{\frac{2}{\pi}}c_4 + \frac{96}{64} - \frac{80}{128}$$

$$= 3070000\sqrt{\frac{2}{\pi}}c_4 + \frac{192-80}{128} = 3070000\sqrt{\frac{2}{\pi}}c_4 + \frac{112}{128}$$

$$c_1 = 383800\ c_4 + \frac{7\sqrt{\pi}}{64\sqrt{2}} \qquad (2.63c)$$

$$2\sqrt{\frac{2}{\pi}}\ c_0 - 4\sqrt{\frac{2}{\pi}}\left(383800\ c_4 + \frac{7\sqrt{\pi}}{64\sqrt{2}}\right) + 6\sqrt{\frac{2}{\pi}}\left(95980\ c_4 + \frac{3\sqrt{\pi}}{64\sqrt{2}}\right) - 8\sqrt{\frac{2}{\pi}}\left(16\ c_4 + \frac{\sqrt{\pi}}{128\sqrt{2}}\right) + 10\sqrt{\frac{2}{\pi}}\ c_4 = 0$$

$$2\sqrt{\frac{2}{\pi}}\ c_0 = 959210\sqrt{\frac{2}{\pi}}\ c_4 + \frac{10}{64} + \frac{8}{128}$$

$$c_0 = 479605\ c_4 + \frac{14\sqrt{\pi}}{128\sqrt{2}} \qquad (2.63d)$$

$$-3.594\times10^{13}c_4 - \frac{1.755\times10^{10}\sqrt{\pi}}{\sqrt{2}} + 1.2201\times10^{14}c_4 + \left[765257.738c_4 + \frac{0.175\sqrt{\pi}}{\sqrt{2}} - 1.5956\ c_4 - 5.361\times10^{5}c_4 - \frac{0.262\sqrt{\pi}}{\sqrt{2}}\right]^3 = 4.79 \qquad (2.64)$$

by(2.64),we have

$c_4 = 0$, and from(2.63)(a), (b), (c) and (d), we get

$c_0 = \frac{14\sqrt{\pi}}{128\sqrt{2}}$, $c_1 = \frac{7\sqrt{\pi}}{64\sqrt{2}}$, $c_2 = \frac{3\sqrt{\pi}}{64\sqrt{2}}$, $c_3 = \frac{\sqrt{\pi}}{128\sqrt{2}}$.

Then, the approximate solution become,

$$y(x) = \frac{14\sqrt{\pi}}{128\sqrt{2}}\cdot 2\sqrt{\frac{2}{\pi}} + \frac{7\sqrt{\pi}}{64\sqrt{2}}\cdot\sqrt{\frac{2}{\pi}}\cdot(8x-4) + \frac{3\sqrt{\pi}}{64\sqrt{2}}\cdot\sqrt{\frac{2}{\pi}}(32x^2-32x+6) + \frac{\sqrt{\pi}}{128\sqrt{2}}\cdot\sqrt{\frac{2}{\pi}}(128x^3-192x^2+80x-8) + 0$$

$$= \frac{28}{128} + \frac{7}{8}x - \frac{7}{16} + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{9}{32} + x^3 - \frac{192}{128}x^2 + \frac{80}{128}x - \frac{8}{128}$$

$= x^3$

2.3.5 The Operational Matrices of Fractional Derivative for Chebyshev Wavelets of Third and Fourth Kinds,[29]:

The third and fourth kind Chebyshev wavelets $\Psi^3_{nm}(t)$ and $\Psi^4_{nm}(t)$ have four argument , k,n∈ N ,m is the order of the polynomial $V^*_m(t)$ or $W^*_m(t)$ and t is the normalized time . They are defined explicitly on the interval [0,1] as

$$\Psi^3_{nm} = \Psi^4_{nm}(t) = \Psi_{nm}(t) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} V^*_n(2^k t - n) & \\ \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} W^*_n(2^k t - n) & , t \in [\frac{n}{2^k}, \frac{n+1}{2^k}] \\ 0 & \text{otherwise.} \end{cases} \quad (2.65a)$$

with ,

m=0,1,…M , n=0,1,…, 2^k -1 (2.65b)

and, the weight function is:

$$\omega^*_{3,n}(2^k t - n) = \sqrt{\frac{(2^k t - n)}{1-(2^k t - n)}} \quad , \quad \omega^*_{4,n}(2^k t - n) = \sqrt{\frac{1-(2^k t - n)}{(2^k t - n)}}$$

2.3.6 Function Approximation for Chebyshev Wavelets of Third and Fourth Kinds,[29]:

The function *f(t)* defined over [0,1] may be expanded in terms for Chebyshev wavelets of third and fourth kinds respectively as follow,

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \Psi^i_{nm}(t) \quad \text{for } i=3,4 \quad (2.66)$$

where, $c_{nm} = \int_0^1 \omega^*_i \, f(t) \, . \, \Psi^i_{nm}(t)\, dt$

$$f(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{m} c_{nm} \Psi^i_{nm}(t)$$

since, C and $\Psi(t)$ are $2^k(M+1)$ matrices defined by

$C = [c_{00}, c_{01}, \ldots. c_{0,M}, \ldots\ldots c_{2^k-1,1}, \ldots\ldots\ldots c_{2^k-1,M}]^T$

$\Psi^i_{nm}(t) = [\Psi^i_{0,0}, \Psi^i_{0,1}, ..\Psi^i_{2^k-1,M}, \ldots\Psi^i_{2^k-1,0}, \Psi^i_{2^k-1,1}, \ldots. \Psi^i_{2^k-1,M}]^T$

2.4 The Relation Between Operational Matrices of Fractional Derivative:

The proposed formulations which express the fractional derivative α>0 operational matrix of shifted third and fourth kind Chybeshev wavelets $D^{\alpha}\Psi^3_{nm}(t)$, $D^{\alpha}\Psi^4_{nm}(t)$ interms of $\Psi^2_{nm}(t)$ and $\Psi^2_{nm-1}(t)$ and first kind Chebyshev wavelets $D^{\alpha}\Psi^1_{\tilde{n}m}(t)$ interms of $\Psi^2_{nm}(t)$ and $\Psi^2_{nm-2}(t)$.

2.4.1 New Relation Between Operational Matrices of Fractional Derivative for $\Psi^2_{nm}(t)$ and $\Psi^3_{nm}(t)$:

From(2.48)(a)and(b), also by

$$U^*_m(t) = \sqrt{\frac{2}{\pi}}\, U_m(t) \quad (2.67a)$$

and, by(2.65)(a)and(b), also by

$V_m^*(t) = \frac{1}{\sqrt{\pi}} V_m\ (t)$ (2.67b)

From(1.47b), we obtain

$V_m(2^k\, t - n)$=$U_m(2^k\, t - n)$-$U_{m-1}(2^k\, t - n)$ (2.68)

with multiplication(2.68) by $\sqrt{\frac{2}{\pi}}$, yield

$$\sqrt{\frac{2}{\pi}}\ V_m(2^k\, t - n) = \sqrt{\frac{2}{\pi}}\ U_m(2^k\, t - n) - \sqrt{\frac{2}{\pi}}\ U_{m-1}(2^k\, t - n)$$

By(2.67)(a)and(b) , we get

$\sqrt{2}V_m^*\left(2^k\, t - n\right)$=$U_m^*\left(2^k\, t - n\right)$-$U_{m-1}^*\left(2^k\, t - n\right)$ (2.69)

with multiplication(2.69) by $\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}$, we have

$$\sqrt{2}\ \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ V_m^*\left(2^k\, t - n\right) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ U_m^*\left(2^k\, t - n\right) - \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ U_{m-1}^*\left(2^k\, t - n\right) \tag{2.70}$$

$2\sqrt{2}\ \Psi_{nm}^3(t) = \Psi_{nm}^2(t) - \Psi_{nm-1}^2(t)$

<u>Theorem(2.4.10) :</u>

Let $\Psi_{nm}^3(t)$ be third kind Chebyshev vector and suppose $\alpha > 0$, then

$$D^{\alpha}\ \Psi_{nm}^3(t) = \frac{2^{\frac{k+3}{2}}}{2\sqrt{2}.\sqrt{\pi}}\ \ D^{\alpha}(U_m^*\left(2^k\, t - n\right)\text{-}U_{m-1}^*\left(2^k\, t - n\right)) = \Delta^{\alpha}\ \Psi_{nm}^3(t)$$

where Δ^{α} is the (m+1) (m+1) operational matrix derivative of order α in the Caputo sense and defined as follow

$$\Delta^{\alpha}= \frac{1}{2\sqrt{2}.}\begin{bmatrix} 0 & & 0 & & 0 \\ . & & . & & . \\ 0 & & 0 & & 0 \\ (\sum_{r=2}^{2} w_{2,0,r}^{\sim} - \sum_{r=2}^{2} w_{2,0,r}^{\approx}) & & (\sum_{r=2}^{2} w_{2,1,r}^{\sim} - \sum_{r=2}^{2} w_{2,1,r}^{\approx}) & \dots & (\sum_{r=2}^{2} w_{2,m,r}^{\sim} - \sum_{r=2}^{2} w_{2,m,r}^{\approx}) \\ \vdots & & \vdots & & \vdots \\ (\sum_{r=2}^{n+1} w_{n+1,0,r}^{\sim} - \sum_{r=2}^{m} w_{m,0,r}^{\approx}) & & (\sum_{r=2}^{n+1} w_{n+1,1,r}^{\sim} - \sum_{r=2}^{m} w_{m,1,r}^{\approx}) & \dots & (\sum_{r=2}^{n+1} w_{n+1,m,r}^{\sim} - \sum_{r=2}^{m} w_{m,m,r}^{\approx}) \\ \vdots & & \vdots & & \vdots \\ (\sum_{r=2}^{m} w_{m,0,r}^{\sim} - \sum_{r=m}^{m} w_{m,0,r}^{\approx}) & & (\sum_{r=2}^{m} w_{m,1,r}^{\sim} - \sum_{r=2}^{m} w_{m,1,r}^{\approx}) & \dots. & (\sum_{r=2}^{m} w_{m,m,r}^{\sim} - \sum_{r=2}^{m} w_{m,m,r}^{\approx}) \end{bmatrix} \tag{2.71}$$

where,

$$w_{m+1,p,r}^{\sim}=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}(-1)^{p+m+2-(\ell+r)}r.\ell(r-1)!\ (m+r)!(p+\ell)!\ \Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m+1-r)!\ 2r!\Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\ \Gamma(r+\ell-\alpha+1)}$$

$$w_{m,p,r}^{\approx} = \frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}r.\ell(-1)^{p+m+1-(\ell+r)}(r-1)!\ (m-1+r)!\,(p+\ell)!\ \Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m-r)!\ 2r!\ \Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\ \Gamma(r+\ell-\alpha+1)}$$

<u>Proof :</u>

from theorem(2.1.7),we have that

$$D^{\alpha}U_m^{*}(2^k t - n)$$

$$= \frac{4}{\sqrt{\pi}} \sum_{r=2}^{m+1}\sum_{p=0}^{m}\sum_{\ell=0}^{p+1} 2^{2(r+\ell)-2} \frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\ (m+r)!\,(p+\ell)!\ \Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m+1-r)!\ 2r!\ \Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\ \Gamma(r+\ell-\alpha+1)}$$

(2.72a)

and, $D^{\alpha}U_{m-1}^{*}(2^k t - n)$

$$= \frac{4}{\sqrt{\pi}} \sum_{r=2}^{m}\sum_{p=0}^{m}\sum_{\ell=0}^{p+1} 2^{2(r+\ell)-2} \frac{r.\ell(-1)^{p+m+1-(\ell+r)}(r-1)!\,(m-1+r)!(p+\ell)!\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m-r)!\,2r!\,\Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\Gamma(r+\ell-\alpha+1)}$$

(2.72b)

from(2.72)(a)and(b), we get

$$D^{\alpha}U_m^{*}(2^k t - n) - D^{\alpha}U_{m-1}^{*}(2^k t - n)$$

$$= \left(\sum_{p=0}^{m}\sum_{r=2}^{m+1} w_{m+1,p,r}^{\sim}\right) U_m^{*}(2^k t - n) - \left(\sum_{p=0}^{m}\sum_{r=2}^{m} w_{m,p,r}^{\sim}\right) U_m^{*}(2^k t - n)$$

(2.73)

by(2.73), we obtain

$$D^{\alpha}\Psi_{nm}^{3}(t)=\frac{2^{\frac{k+3}{2}}}{2\sqrt{2}.\sqrt{\pi}} \sum_{p=0}^{m}\left[\sum_{r=2}^{m+1} w_{m+1,p,r}^{\sim} - \sum_{r=2}^{m} w_{m,p,r}^{\sim}\right] U_m^{*}(2^k t - n)\,.$$

for $n \geq \lceil\alpha\rceil$, we have

$$D^{\alpha}\Psi_{nm}^{3}(t)=\frac{1}{2\sqrt{2}}\left[\left(\sum_{r=2}^{m+1} w_{m+1,0,r}^{\sim} - \sum_{r=2}^{m} w_{m,0,r}^{\sim}\right), \left(\sum_{r=2}^{m+1} w_{m+1,1,r}^{\sim} - \sum_{r=2}^{m} w_{m,1,r}^{\sim}\right), \ldots, \left(\sum_{r=2}^{m+1} w_{m+1,m,r}^{\sim} - \sum_{r=2}^{m} w_{m,m,r}^{\sim}\right.\right]. \Psi_{nm}^{3}(t)$$

(2.74)

and, $D^{\alpha}\Psi_{nm}^{3}(t) = \frac{2^{\frac{k+3}{2}}}{2\sqrt{2}.\sqrt{\pi}}\left(D^{\alpha}U_m^{*}(2^k t - n) - D^{\alpha}U_{m-1}^{*}(2^k t - n)\right)$

$$=\frac{1}{2\sqrt{2}}\,[\,0,0,\ldots,0\,]\ \Psi_{nm}^{3}(t) \qquad n > \lceil\alpha\rceil$$

(2.75)

2.4.2 New Relation between Operational Matrices of Fractional Derivative for $\Psi_{nm}^{2}(t)$ and $\Psi_{nm}^{4}(t)$:

from(2.48)(a)and(b), also by(2.67a) and, by(2.65a) with

$$W_m^{*}(t) = \frac{1}{\sqrt{\pi}} W_m(t) \qquad (2.76)$$

by(1.47c),we obtain

$$W_m(2^k t - n) = U_m(2^k t - n) + U_{m-1}(2^k t - n) \qquad (2.77)$$

with multiplication(2.77)by $\sqrt{\frac{2}{\pi}}$, yield

$$\sqrt{\frac{2}{\pi}}\, W_m(2^k t - n) = \sqrt{\frac{2}{\pi}}\, U_m(2^k t - n) + U_{m-1}(2^k t - n)$$

from(2.67a)and(2.76),we get

$$\sqrt{2}\, W_m^{*}(2^k t - n) = U_m^{*}(2^k t - n) + U_{m-1}^{*}(2^k t - n) \qquad (2.78)$$

with multiplication(2.78) by $\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}$,we have

$$\sqrt{2}\cdot\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ W_m^*\left(2^k t-n\right)=\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ U_m^*\left(2^k t-n\right)+\ \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ U_{m-1}^*\left(2^k t-n\right) \tag{2.79}$$

$$2\sqrt{2}\cdot\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\ W_m^*\left(2^k t-n\right)=\Psi_{nm}^2(t)-\Psi_{nm-1}^2(t)$$

$$2\sqrt{2}\ \Psi_{nm}^4(t)=\Psi_{nm}^2(t)-\Psi_{nm-1}^2(t)$$

Theorem (2.4.11):

Let $\varphi_{nm}^4(t)$ be fourth kind Chebyshev vector and suppose $\alpha>0$. Then

$$D^\alpha\Psi_{nm}^4(t)=\frac{2^{\frac{k+3}{2}}}{2\sqrt{2}\cdot\sqrt{\pi}}D^\alpha\left(U_m^*\left(2^k t-n\right)+U_{m-1}^*\left(2^k t-n\right)\right)$$
$$=\Delta^\alpha\Psi_{nm}^4(t)$$

where Δ^α is the (m+1) (m+1) operational matrix derivative of order α in the Caputo sense and defined as follow:

$$\Delta^\alpha = \frac{1}{2\sqrt{2}.}\begin{bmatrix} 0 & 0 & 0 \\ . & . & . \\ 0 & 0 & 0 \\ \left(\sum_{r=2}^{2} w_{2,0,r}^{\sim}-\sum_{r=2}^{2} w_{2,0,r}^{\approx}\right) & \left(\sum_{r=2}^{2} w_{2,1,r}^{\sim}-\sum_{r=2}^{2} w_{2,1,r}^{\approx}\right)\ \cdots & \left(\sum_{r=2}^{2} w_{2,m,r}^{\sim}-\sum_{r=2}^{2} w_{2,m,r}^{\approx}\right) \\ \vdots & \vdots & \vdots \\ \left(\sum_{r=2}^{n+1} w_{n+1,0,r}^{\sim}-\sum_{r=2}^{m} w_{m,0,r}^{\approx}\right) & \left(\sum_{r=2}^{n+1} w_{n+1,1,r}^{\sim}-\sum_{r=2}^{m} w_{m,1,r}^{\approx}\right)\ \cdots & \left(\sum_{r=2}^{n+1} w_{n+1,m,r}^{\sim}-\sum_{r=2}^{m} w_{m,m,r}^{\approx}\right) \\ \vdots & \vdots & \vdots \\ \left(\sum_{r=2}^{m} w_{m,0,r}^{\sim}-\sum_{r=m}^{m} w_{m,0,r}^{\approx}\right) & \left(\sum_{r=2}^{m} w_{m,1,r}^{\sim}-\sum_{r=2}^{m} w_{m,1,r}^{\approx}\right)\ \cdots. & \left(\sum_{r=2}^{m} w_{m,m,r}^{\sim}-\sum_{r=2}^{m} w_{m,m,r}^{\approx}\right) \end{bmatrix} \tag{2.80}$$

where,

$$w_{m+1,p,r}^{\sim}=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\ \ 2^{2(r+\ell)-2}\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m+1-r)!\,2r!\,\Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\,\Gamma(r+\ell-\alpha+1)}$$

$$w_{m,p,r}^{\approx}=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}r.\ell(-1)^{p+m+1-(\ell+r)}(r-1)!\,(m-1+r)!(p+\ell)!\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m-r)!\,2r!\Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\,\Gamma(r+\ell-\alpha+1)} \tag{2.81}$$

<u>2.4.3 New Relation Between Operational Matrices of Fractional Derivative for $\Psi_{nm}^2(t)$, $\Psi_{nm}^3(t)$ and $\Psi_{nm}^4(t)$:</u>

from(2.48)(a)and(b), also by(2.67a)

and, by(2.65)(a),(b) with(2.67b),(2.76),and (1.47a), we get

$$2U_m\left(2^k t-n\right)=V_m\left(2^k t-n\right)+W_m\left(2^k t-n\right) \tag{2.82}$$

with multiplication(1.47a)by $\frac{1}{\sqrt{\pi}}$, we obtain

$\sqrt{2}\cdot\sqrt{\frac{2}{\pi}}\,U_m(2^k t-n)=\frac{1}{\sqrt{\pi}}V_m(2^k t-n)+\frac{1}{\sqrt{\pi}}W_m(2^k t-n)$

from(2.67)(a),(b),we have

$\sqrt{2}\,.\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}U_m^*(2^k t-n)=\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}V_m^*(2^k t-n)+\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}W_m^*(2^k t-n)$

$$\frac{2^{\frac{k+3}{2}}}{\sqrt{2}\sqrt{\pi}}U_m^*(2^k t-n)=\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}V_m^*(2^k t-n)+\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}W_m^*(2^k t-n) \qquad (2.83)$$

$\frac{1}{\sqrt{2}}\Psi_{nm}^2(t)=\Psi_{nm}^3(t)+\Psi_{nm}^4(t)$

Theorem (2.4.12):

Let $\Psi_{nm}^2(t)$,$\Psi_{nm}^3(t)$and $\Psi_{nm}^4(t)$ are shifted second, third and fourth kind Chebyshev vector respectively and suppose $\alpha>0$, Then:

$$D^\alpha\Psi_{nm}^2(t)=\Delta^\alpha\Psi_{nm}^2(t)$$

$$=\frac{2^{\frac{k+1}{2}}}{2\sqrt{2}\cdot\sqrt{\pi}}D^\alpha\left(V_m^*(2^k t-n)+W_m^*(2^k t-n)\right)$$

$$=\frac{1}{2\sqrt{2}\cdot}(\Delta^\alpha\varphi_{nm}^3(t)+\Delta^\alpha\Psi_{nm}^4(t)) \qquad (2.84)$$

Where Δ^α is the $(m+1)(m+1)$ operational Matrix derivative of order α in the Caputo derivative.

2.4.4 New Relation between Operational Matrices of Fractional Derivative for $\Psi_{\tilde{n}m}^1(t)$ and $\Psi_{nm}^2(t)$:

from(2.32)(a)and(b), also by(2.48a),(2.67a),and by(1.44),we get

$$2\,T_m(2^k t-n)=U_m(2^k t-n)+U_{m-2}(2^k t-n) \qquad m=2,3,\cdots \qquad (2.85)$$

with multiplication(2.85) by $\sqrt{\frac{2}{\pi}}$, we get

$2\sqrt{\frac{2}{\pi}}\,T_m(2^k t-n)=\sqrt{\frac{2}{\pi}}\,U_m(2^k t-n)-\sqrt{\frac{2}{\pi}}\,U_{m-2}(2^k t-n)$

from(2.32b)and(2.67a), we get

$$2\,T_m^*(2^k t-n)=U_m^*(2^k t-n)+U_{m-2}^*(2^k t-n) \qquad (2.86)$$

with multiplication(2.86) by $\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}$, we get

$$2\cdot\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\,T_m^*(2^k t-n)=\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\,U_m^*(2^k t-n)+\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\,U_{m-2}^*(2^k t-n) \qquad (2.87)$$

$$c_m=\begin{cases}1 & m=0\\ \sqrt{2} & m\neq 0\end{cases}$$

$\frac{4\sqrt{2}}{c_m}\,\Psi_{\tilde{n}m}^1(t)=\Psi_{nm}^2(t)-\Psi_{nm-2}^2(t)\,,\quad m=2,3,\cdots$

Theorem (2.4.13):

Let $\Psi_{\tilde{n}m}^1(t)$ be fourth kind Chebyshev vector and suppose $\alpha>0$, then

$$D^{\alpha}\Psi^{1}_{\tilde{n}m}(t)(t)=\frac{c_m 2^{\frac{k+3}{2}}}{4\sqrt{2}\cdot\sqrt{\pi}}D^{\alpha}\left(U^{*}_{m}\left(2^{k}t-n\right)+U^{*}_{m-1}\left(2^{k}t-n\right)\right)$$
$$=\Delta^{\alpha}\Psi^{2}_{nm}(t)$$

where Δ^{α} is the $(m+1)(m+1)$ operational Matrix derivative of order α in the Caputo sense and defined as follow:

$$\Delta^{\alpha} = \frac{C_m}{4\sqrt{2}.}\begin{bmatrix} 0 & 0 & 0 \\ . & . & . \\ 0 & 0 & 0 \\ (\sum_{r=2}^{2} w^{\sim}_{2,0,r}-\sum_{r=2}^{2} w^{\approx}_{2,0,r}) & (\sum_{r=2}^{2} w^{\sim}_{2,1,r}-\sum_{r=2}^{2} w^{\approx}_{2,1,r}) & \dots(\sum_{r=2}^{2} w^{\sim}_{2,m,r}-\sum_{r=2}^{2} w^{\approx}_{2,m,r}) \\ \vdots & \vdots & \vdots \\ (\sum_{r=2}^{n+1} w^{\sim}_{n+1,0,r}-\sum_{r=2}^{n-1} w^{\approx}_{n-1,0,r}) & (\sum_{r=2}^{n+1} w^{\sim}_{n+1,1,r}-\sum_{r=2}^{n-1} w^{\approx}_{n-1,1,r})\dots & (\sum_{r=2}^{n+1} w^{\sim}_{n+1,m,r}-\sum_{r=\lceil\alpha\rceil+1}^{n-1} w^{\approx}_{n-1,m,r}) \\ \vdots & \vdots & \vdots \\ (\sum_{r=2}^{m} w^{\sim}_{m,0,r}-\sum_{r=2}^{m} w^{\approx}_{m,0,r}) & (\sum_{r=2}^{m} w^{\sim}_{m,1,r}-\sum_{r=2}^{m} w^{\approx}_{m,1,r}) \quad \dots & (\sum_{r=2}^{m} w^{\sim}_{m,m,r}-\sum_{r=2}^{m} w^{\approx}_{m,m,r}) \end{bmatrix}$$

(2.88)

where,

$$w^{\sim}_{m+1,p,r}=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}\, r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m+1-r)!\,2r!\,\Gamma(r-\alpha).(p+1-\ell)!\;\,2\ell!\,\Gamma(r+\ell-\alpha+1)}$$

(2.89a)

$$w^{\approx}_{m-2,p,r}=\frac{4}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}r.\ell(-1)^{p+m-(\ell+r)}(r-1)!\;(m-2+r)!\,(p+\ell)!\;\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m-1-r)!\;2r!\;\Gamma(r-\alpha).(p+1-\ell)!\;\;2\ell!\;\Gamma(r+\ell-\alpha+1)}$$

(2.89b)

<u>**Proof**</u>:

from(2.53),we have

$$D^{\alpha}U^{*}_{m}\left(2^{k}t-n\right)=\frac{4}{\sqrt{\pi}}\sum_{r=2}^{m+1}\sum_{p=0}^{m}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m+1-r)!\,2r!\Gamma(r-\alpha).(p+1-\ell)!\;\,2\ell!\,\Gamma(r+\ell-\alpha+1)}$$

(2.90a)

and, $D^{\alpha}U^{*}_{m-2}\left(2^{k}\,t-\ n\right)=$

$$\frac{4}{\sqrt{\pi}}\sum_{r=2}^{m-2}\sum_{p=0}^{M}\sum_{\ell=0}^{p+1}\frac{2^{2(r+\ell)-2}r.\ell(-1)^{p+m-(\ell+r)}(r-1)!\,(m-2+r)!(p+\ell)!\,\Gamma\left(r-\alpha+\ell-\frac{1}{2}\right)}{(m-1-r)!\,2r!\,\Gamma(r-\alpha).(p+1-\ell)!\ \ 2\ell!\,\Gamma(r+\ell-\alpha+1)}$$

(2.90b)

from(2.90)(a)and(b),we get

$D^{\alpha}U^{*}_{m}\left(2^{k}\,t-\ n\right)-D^{\alpha}U^{*}_{m-2}\left(2^{k}\,t-\ n\right)$

$=\left(\sum_{p=0}^{m}\sum_{r=2}^{m+1}w^{\sim}_{m+1\,,p\,,r}\right)U^{*}_{m}\left(2^{k}t-n\right)-\left(\sum_{p=0}^{m}\sum_{r=2}^{m-1}w^{\sim}_{m-1\,,p\,,r}\right)U^{*}_{m}\left(2^{k}t-n\right).$

Thus,

$D^{\alpha}\Psi^{1}_{\tilde{n}m}(t)=\frac{C_m 2^{\frac{k+3}{2}}}{4\sqrt{2}.\sqrt{\pi}}\ \sum_{p=0}^{m}\left[\sum_{r=2}^{m+1}w^{\sim}_{m+1\,,p\,,r}-\sum_{r=2}^{m-1}w^{\sim}_{m-1\,,p\,,r}\right]\ U^{*}_{m}\ \left(2^{k}t-n\right).$

Then

$$D^{\alpha}\Psi^{1}_{\tilde{n}m}(t)=\frac{C_m 2^{\frac{k+3}{2}}}{4\sqrt{2}.\sqrt{\pi}}\left[\left(\sum_{r=2}^{m+1}w^{\sim}_{m+1,0,r}-\sum_{r=2}^{m-1}w^{\sim}_{m-1\,,0\,,r}\right),\left(\sum_{r=2}^{m+1}w^{\sim}_{m+1,1,r}-\sum_{r=2}^{m-1}w^{\sim}_{m-1\,,1\,,r}\right),\ldots,\left(\sum_{r=2}^{m+1}w^{\sim}_{m+1,m,r}-\sum_{r=2}^{m-1}w^{\sim}_{m-1\,,m\,,r}\right)\right].\Psi^{2}_{nm}(t)$$

(2.91)

and, $D^{\alpha}\Psi^{1}_{\tilde{n}m}(t)=\frac{C_m 2^{\frac{k+3}{2}}}{4\sqrt{2}.\sqrt{\pi}}\left(D^{\alpha}U^{*}_{m}\left(2^{k}\,t-\ n\right)-D^{\alpha}U^{*}_{m-1}\left(2^{k}\,t-\ n\right)\right)$

$=\frac{C_m}{4\sqrt{2}}\,[\ 0,0,\ldots,0\ ]\,\Psi^{2}_{nm}(t)\qquad n<\lceil\alpha\rceil$

(2.92)

CHAPTER THREE

Fractional Operational Matrices of Fractional Derivative for Solving (MFNDE) With MIXED Boundary Conditions

2.1 Introduction:

3.1 Introduction:

The multi-fractional order nonlinear differential equation (MFNDE) arise in modeling processes in applied sciences, as in physics, engineering, chemistry[18]. Other sciences can be described very successfully by models using mathematical tools from fractional calculus, and concepts of fractional polynomials and fractional operational matrices,[11],[17],[19],[33],[50].

The fractional operational matrices are usually difficult to be formulated analytically, so, it is required to obtain an efficient approximate solution of (MFNDE).

This chapter consists mainly of three sections. In section (3.2), which is termed as shifted first, third and fourth kinds of fractional order chebyshev polynomials. In section (3.3), the operational matrix of fractional derivative for shifted first, third and fourth kinds chebyshev polynomials is presented. In section (3.4), the couple fractional order for shifted chebyshev polynomials.

The same as [17, 33], we present some kinds of fractional order chebyshev polynomials.

3.2 Some Kinds of Fractional Order Chebyshev polynomials:

We introduce the fractional order chebyshev polynomials of first, third and fourth kinds by changing the variable $x = x^{\alpha}$which $\alpha > 0$ in (1.11), (1.41b)and(1.42b)as follows:

1. $T_n^*(x^{\alpha})$ as $\bar{T}_n^{\alpha}(x)$.
2. $V_n^*(x^{\alpha})$ as $\bar{V}_n^{\alpha}(x)$.
3. $W_n^*(x^{\alpha})$ as $\bar{W}_n^{\alpha}(x)$.

From the recurrence relation of the shifted Chebyshev polynomials of all above kinds, we can be obtain with the following recurrence formula :

i. $\bar{T}_{n+1}^{\alpha}(x) = 2\,x^{\alpha}\,\bar{T}_n^{\alpha}(x) - \bar{T}_{n-1}^{\alpha}(x)\,, \qquad n = 1\,,2\,,\cdots$ (3.1)

where $\bar{T}_0^{\alpha}(x) = 1\,,\,\bar{T}_1^{\alpha}(x) = 2x^{\alpha} - 1$

ii. $\bar{V}_{n+1}^{\alpha}(x) = 2(2\,x^{\alpha} - 1)\bar{V}_n^{\alpha}(x) - \bar{V}_{n-1}^{\alpha}(x)\,, \quad n = 2\,,3\,,\cdots$ (3.2)

where $\bar{V}_0^{\alpha}(x) = 1\,,\,\bar{V}_1^{\alpha}(x) = 4x^{\alpha} - 3$

iii. $\bar{W}_{n+1}^{\alpha}(x) = 2(2\,x^{\alpha} - 1)\bar{W}_{n}^{\alpha}(x) - \bar{W}_{n-1}^{\alpha}(x)\,, \quad n = 1\,,2\,,\cdots$ (3.3)

where $\bar{W}_0^{\alpha}(x) = 1$, $\bar{W}_1^{\alpha}(x) = 2x^{\alpha} + 1$

<u>Lemma(3.2.1)</u>:

1. $\bar{T}_n^{\alpha}(x)$ are orthogonal function with respect to the weight function $\omega_{1,\alpha}^{*}(x) = \frac{1}{x^{\alpha}\sqrt{x^{-2\alpha}-1}}$,and we have

$$\int_0^1 \bar{T}_n^{\alpha}(x)\cdot\bar{T}_m^{\alpha}(x)\cdot\,\omega_{1,\alpha}^{*}(x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{\pi c_i}{4\cdot\alpha} & n = m \end{cases} \tag{3.4}$$

where, $c_i = \begin{cases} 2 & if \quad i = 0 \\ 1 & if \quad i \geq 1 \end{cases}$.

2. $\bar{V}_n^{\alpha}(x)$ are orthogonal with respect to the weight function $\omega_{3,\alpha}^{*}(x) = x^{\alpha-1}\sqrt{(x^{-\alpha}-1)^{-1}}$,and we have

$$\int_0^1 \bar{V}_n^{\alpha}(x)\cdot V_m^{\alpha}(x)\cdot\,\omega_{3,\alpha}^{*}(x)\,dx = \begin{cases} \frac{\pi}{2\cdot\alpha} & if \quad n = m \\ 0 & if \quad n \neq m \end{cases} \tag{3.5}$$

3. $\bar{W}_n^{\alpha}(x)$ are orthogonal with respect to the weight function $\omega_{4,\alpha}^{*}(x) = x^{\alpha-1}\sqrt{x^{-\alpha}-1}$, and we have

$$\int_0^1 \bar{W}_n^{\alpha}(x)\cdot\bar{W}_m^{\alpha}(x)\cdot\,\omega_{4,\alpha}^{*}(x)\,dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{\pi}{2\cdot\alpha} & if \quad n = m \end{cases} \tag{3.6}$$

<u>Proof:</u>

1. By taking $t = x^{\alpha}$ and $dt = \alpha x^{\alpha-1}dx$, Substituting these valued in

$$\int_0^1 T_n^{*}(t)\cdot T_m^{*}(t)\cdot\,\omega_1^{*}(t)dt = \begin{cases} 0 & if \quad n \neq m \\ \frac{\pi c_i}{4} & if \quad n = m \end{cases}$$

we get,

$$\int_0^1 T_n^{*}(x^{\alpha})\cdot T_m^{*}(x^{\alpha})\cdot\,\omega_{1,\alpha}^{*}(x)\,\cdot\alpha x^{\alpha-1}dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{\pi c_i}{4} & if \quad n = m \end{cases} \tag{3.7}$$

$$\alpha\int_0^1 \bar{T}_n^{\alpha}(x)\cdot\bar{T}_m^{\alpha}(x)\cdot\frac{x^{\alpha-1}}{x^{\alpha}\sqrt{x^{-2\alpha}-1}}\,dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{\pi c_i}{4} & if \quad n = m \end{cases}$$

$$\int_0^1 \bar{T}_n^{\alpha}(x)\cdot\bar{T}_m^{\alpha}(x)\cdot\,\omega_{1,\alpha}^{*}(x)\,dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{\pi c_i}{4\cdot\alpha} & if \quad n = m \end{cases} \tag{3.8}$$

2. for taking $t = x^{\alpha}$ and $dt = \alpha x^{\alpha-1}dx$, substituting these valued in

$$\int_0^1 V_n^{*}(t)\cdot V_m^{*}(t)\cdot\omega_3^{*}(t)dt = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2\cdot\alpha} & n = m \end{cases}$$

we obtain,

$$\int_0^1 V_n^{*}(x^{\alpha})\cdot V_m^{*}(x^{\alpha})\cdot\,\omega_{3,\alpha}^{*}(x)\cdot\alpha x^{\alpha-1}dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{\pi}{2} & if \quad n = m \end{cases} \tag{3.9}$$

$$\alpha\int_0^1 \bar{V}_n^\alpha(x)\cdot\bar{V}_m^\alpha(x)\cdot\sqrt{\frac{x^\alpha}{1-x^\alpha}}\cdot x^{\alpha-1}\,dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases}$$

$$\alpha\int_0^1 \bar{V}_n^\alpha(x)\cdot\bar{V}_m^\alpha(x)\cdot\sqrt{\frac{1}{x^{-\alpha}-1}}\cdot x^{\alpha-1}\,dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases}$$

$$\alpha\int_0^1 \bar{V}_n^\alpha(x)\cdot\bar{V}_m^\alpha(x)\cdot x^{\alpha-1}\cdot\sqrt{(x^{-\alpha}-1)^{-1}}\,dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases}$$

Then

$$\int_0^1 \bar{V}_n^\alpha(x)\cdot V_m^\alpha(x)\cdot\ \omega_{3,\alpha}^*(x)\ dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2\cdot\alpha} & if \quad n=m\end{cases}$$

3. now for taking $t=x^\alpha$ and $dt=\alpha x^{\alpha-1}dx$, substituting these valued in

$$\int_0^1 W_n^*(t)\cdot W_m^*(t)\cdot\ \omega_4^*(t)dt=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases}$$

we get,

$$\int_0^1 W_n^*(x^\alpha)\cdot W_m^*(x^\alpha)\cdot\ \omega_{4,\alpha}^*(x)\cdot\alpha x^{\alpha-1}dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases} \qquad (3.10)$$

$$\int_0^1 \bar{W}_n^\alpha(x)\cdot\bar{W}_m^\alpha(x)\cdot\frac{x^{\frac{1}{2}\alpha}}{x^{\frac{1}{2}\alpha}}\sqrt{x^{-\alpha}-1}\cdot\alpha x^{\alpha-1}\,dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases}$$

$$\alpha\int_0^1 \bar{W}_n^\alpha(x)\cdot\bar{W}_m^\alpha(x)\cdot x^{\alpha-1}\cdot\sqrt{x^{-\alpha}-1}\,dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2} & if \quad n=m\end{cases}$$

$$\int_0^1 \bar{W}_n^\alpha(x)\cdot\bar{W}_m^\alpha(x)\cdot\ \omega_{4,\alpha}^*(x)\ dx=\begin{cases}0 & if \quad n\neq m\\ \frac{\pi}{2\cdot\alpha} & if \quad n=m\end{cases}$$

<u>lemma(3.2.2):</u>

1)The fractional-order first kind Chebyshev function $\bar{T}_n^\alpha(x)$, has precisely n zeros in the form:

$$t_m=\left(\frac{1+\cos\frac{\left(m-\frac{1}{2}\right)\pi}{n}}{2}\right)^{\frac{1}{\alpha}} \qquad m=1\,,2\,,\cdots,n\,. \qquad (3.11)$$

2)The fractional-order third kind Chebyshev function $\bar{V}_n^\alpha(x)$, has precisely n zeros in the form:

$$t_m=\left(\frac{1+\cos\frac{\left(m-\frac{1}{2}\right)\pi}{n+\frac{1}{2}}}{2}\right)^{\frac{1}{\alpha}} \qquad m=1\,,2\,,\cdots,n\,. \qquad (3.12)$$

3)The fractional-order forth kind Chebyshev function $\bar{W}_n^{\alpha}(x)$,has precisely n zeros in the form:

$$t_m = \left(\frac{1+\cos\left[\frac{m\pi}{n+\frac{1}{2}}\right]}{2} \right)^{\frac{1}{\alpha}} \qquad m = 1, 2, \ldots, n . \tag{3.13}$$

***Proof*:**

1)The shifted Chebyshev polynomial of first kind $T_n^*(x)$ has n zeros

$$x_m = \left(\frac{1+\cos\left[\frac{\left(m-\frac{1}{2}\right)\pi}{n}\right]}{2} \right) \qquad m = 1, 2, \cdots, n .$$

so, $T_n^*(x)$ can be written as

$$T_n^*(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \tag{3.14a}$$

changing of variable $x = t^{\alpha}$ in(3.14a),yields

$$\bar{T}_n^{\alpha}(t) = (t^{\alpha} - x_1)(t^{\alpha} - x_2) \cdots (t^{\alpha} - x_n) \tag{3.14b}$$

so, the zeros of $\bar{T}_n^{\alpha}(t)$ are,

$$t_m = (x_m)^{\frac{1}{\alpha}} \qquad m = 1, 2, \cdots, n \tag{3.15}$$

2)The shifted third kind Chebyshev polynomial $V_n^*(x)$ has n zeros

$$x_m = \left(\frac{1+\cos\frac{\left(m-\frac{1}{2}\right)\pi}{n+\frac{1}{2}}}{2} \right) \qquad m = 1, 2, \cdots, n .$$

so, $V_n^*(x)$ can be written as

$$V_n^*(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \tag{3.16a}$$

changing of variable $x = t^{\alpha}$ in(3.16a),yields

$$\bar{V}_n^{\alpha}(t) = (t^{\alpha} - x_1)(t^{\alpha} - x_2) \cdots (t^{\alpha} - x_n) \tag{3.16b}$$

so, the zeros of $\bar{V}_n^{\alpha}(t)$ are

$$t_m = (x_m)^{\frac{1}{\alpha}} \qquad m = 1, 2, \cdots, n \tag{3.17}$$

3)The shifted fourth kind Chebyshev polynomial $W_n^*(x)$ has n zeros:
Then $W_n^*(x)$ can be written as

$$W_n^*(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \tag{3.18a}$$

changing of variable $x = t^{\alpha}$ in (3.18a),yields

$$\bar{W}_n^{\alpha}(t) = (t^{\alpha} - x_1)(t^{\alpha} - x_2) \cdots (t^{\alpha} - x_n) \tag{3.18b}$$

so, the zeros of $\bar{W}_n^{\alpha}(t)$ are , $t_m = (x_m)^{\frac{1}{\alpha}} \qquad m = 1, 2, \cdots, n.$ (3.19)

Remark(3.2.3):

a- For any function $f \in L^2_{w_{\alpha_1}}$ we write

$$f = \sum_{k=0}^{\infty} f_k \bar{T}_k^{\alpha}(x)\,, \quad with \quad f_k = \frac{\langle f\,, \bar{T}_k^{\alpha}\rangle_{w_{\alpha_1}}}{\|\bar{T}_k^{\alpha}\|_{w_{\alpha_1}}^2} \tag{3.20}$$

where, f_k is the expansion coefficients associated with the family $\{\bar{T}_k^{\alpha}\}$.

b- For any function $f \in L_{w_{\alpha_3}}^2$ we write

$$f = \sum_{k=0}^{\infty} f_k \bar{V}_k^{\alpha}(x)\,, \quad with \quad f_k = \frac{\langle f\,, \bar{V}_k^{\alpha}\rangle_{w_{\alpha_3}}}{\|\bar{V}_k^{\alpha}\|_{w_{\alpha_3}}^2} \tag{3.21}$$

where, f_k is the expansion coefficients associated with the family $\{\bar{V}_k^{\alpha}\}$.

c- For any function $f \in L_{w_{\alpha_4}}^2$ we write

$$f = \sum_{k=0}^{\infty} f_k \bar{W}_k^{\alpha}(x)\,, \quad with \quad f_k = \frac{\langle f\,, \bar{W}_k^{\alpha}\rangle_{w_{\alpha_4}}}{\|\bar{W}_k^{\alpha}\|_{w_{\alpha_4}}^2} \tag{3.22}$$

where, f_k are the expansion coefficients associated with the family $\{\bar{W}_k^{\alpha}\}$.

3.3 The Operational Matrix of Fractional Derivative:

Let,

i- $\overline{T_\alpha}(x) = \{\bar{T}_0^{\alpha}(x), \bar{T}_1^{\alpha}(x), \cdots, \bar{T}_N^{\alpha}(x)\}^T$.

ii- $\bar{V_\alpha}(x) = \{\bar{V}_0^{\alpha}(x), \bar{V}_1^{\alpha}(x), \cdots, \bar{V}_N^{\alpha}(x)\}^T$

iii- $\overline{W_\alpha}(x) = \{\bar{W}_0^{\alpha}(x), \bar{W}_1^{\alpha}(x), \cdots, \bar{W}_N^{\alpha}(x)\}^T$.

and $X_\alpha(x) = \{1, x^{\alpha}, x^{2\alpha}, \cdots, x^{N\alpha},\}^T$

we obtain,

a- $\overline{T_\alpha}(x) = F^{(1)} X_\alpha$. (3.23)

or $\bar{T}_i^{\alpha}(x) = \sum_{j=0}^{N} f_{ij}^{(1)} x^{j\alpha} \qquad i = 0\,, 1\,, \cdots, N$.

b- $\bar{V_\alpha}(x) = F^{(3)} X_\alpha$. (3.24)

or $\bar{V}_i^{\alpha}(x) = \sum_{j=0}^{N} f_{ij}^{(3)} x^{j\alpha} \qquad i = 0\,, 1\,, \cdots, N$.

c- $\overline{W_\alpha}(x) = F^{(4)} X_\alpha$. (3.25)

or $\bar{W}_i^{\alpha}(x) = \sum_{j=0}^{N} f_{ij}^{(4)} x^{j\alpha} \qquad i = 0\,, 1\,, \cdots, N$.

The fractional derivative of order λ of the vector $T_\alpha(x)$, $V_\alpha(x)$ and $W_\alpha(x)$ can be expressed by,

1. $D^{\lambda}\overline{T_\alpha}(x) \cong \Delta^{\lambda}\overline{T_\alpha}(x)$ (3.26)

2. $D^{\lambda}\bar{V_\alpha}(x) \cong \Delta^{\lambda}\bar{V_\alpha}(x)$ (3.27)

3. $D^{\lambda}\overline{W_\alpha}(x) \cong \Delta^{\lambda}\overline{W_\alpha}(x)$ (3.28)

where Δ^{λ} is the $(n+1)(n+1)$ operational matrix of fractional derivative.

lemma (3.3.3),[33] :

Let, $k = \begin{cases} \text{the largest integer such that } k\alpha < \lceil\lambda\rceil & for \quad \alpha \in N_0 \\ 0 & for\ \alpha \notin N\ and\ \alpha < \lceil\lambda\rceil \end{cases}$

(3.29a)

Then, we have $D^{\lambda}X_{\alpha}(x) \cong \bar{\Delta}^{\lambda}.X_{\alpha}^{\lambda}(x)$
where $\bar{\Delta}^{\lambda}$ is the following $(n+1)(n+1)$ diagonal matrix

$$\bar{\Delta}^{\lambda} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ 0 & \cdots & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & \cdots & 0 \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}$$

(3.29b)

$X_{\alpha}^{\lambda}(x) = \left[0, \cdots, 0, x^{(k+1)\alpha-\lambda}, x^{(k+2)\alpha-\lambda}, \cdots, x^{N\alpha-\lambda}\right]^T$

lemma (3.3.4),[33]:

we have, $X_{\alpha}^{\lambda}(x) \cong B\bar{U}_i^{\alpha}(x)$
where $B = (b_{ij})$ is the following $(n+1)(n+1)$ matrix,

$$b_{ij} = \begin{cases} 0 & \begin{cases} i = 0,1,\cdots,k \\ j = 0,1,\cdots,N \end{cases} \\ \frac{\sqrt{\pi}}{\tau} \cdot \sum_{\ell=0}^{j-1} f_{j\ell} \frac{\Gamma\left(i - \frac{\lambda}{\alpha} + l + \frac{5}{2}\right)}{\Gamma\left(i - \frac{\lambda}{\alpha} + l + 3\right)} - \frac{\Gamma\left(i - \frac{\lambda}{\alpha} + l + \frac{7}{2}\right)}{\Gamma\left(i - \frac{\lambda}{\alpha} + l + 4\right)} & \begin{cases} i = k+1, k+2, \cdots, N \\ j = 0,1,\cdots,N \end{cases} \end{cases}$$

Theorem (3.3.13):

a- We have, $X_{\alpha}^{\lambda}(x) \cong D\bar{T}_i^{\alpha}(x)$

where, $G = (g_{ij})$ is the following $(n+1)(n+1)$ matrix .

$$g_{ij} = \begin{cases} 0 & \begin{cases} i = 0,1,\cdots,k \\ j = 0,1,\cdots,N \end{cases} \\ \frac{2}{\pi \cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \frac{\Gamma\left(\frac{1}{2}(i+\ell) - \frac{\lambda}{2\alpha} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(i+\ell) - \frac{\lambda}{2\alpha} + 1\right)} & \begin{cases} i = k+1, k+2, \cdots, N \\ j = 0,1,\cdots,N \end{cases} \end{cases}$$

(3.30)

b- We have, $X_{\alpha}^{\lambda}(x) \cong H \cdot \bar{V}_i^{\alpha}(x)$

where, $H = (h_{ij})$ is the following $(n+1)(n+1)$ matrix .

$$h_{ij} = \begin{cases} 0 & \begin{cases} i = 0,1,\cdots,k \\ j = 0,1,\cdots,N \end{cases} \\ \frac{-4}{\sqrt{\pi}} \sum_{\ell=0}^{j} f_{i\ell} \cdot \frac{\Gamma\left(i + l - \frac{\lambda}{\alpha} - \frac{1}{\alpha} + \frac{7}{2}\right)}{\Gamma\left(i + l - \frac{\lambda}{\alpha} - \frac{1}{\alpha} + 3\right)} & \begin{cases} i = k+1, k+2, \cdots, N \\ j = 0,1,\cdots,N \end{cases} \end{cases}$$

(3.31)

c- We have, $X_\alpha^\lambda(x) \cong R \cdot \bar{W}_i^\alpha(x)$

where, $R = (r_{ij})$ is the following $(n+1)(n+1)$ matrix .

$$r_{ij} = \begin{cases} 0 & \begin{cases} i = 0,1,\cdots,k \\ j = 0,1,\cdots,N \end{cases} \\ \frac{-1}{\sqrt{\pi}} \cdot \sum_{\ell=0}^{j} f_{i\ell} \cdot \frac{\Gamma\left(i+l-\frac{\lambda}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(i+l-\frac{\lambda}{\alpha}+2\right)} & \begin{cases} i = k+1, k+2, \cdots, N \\ j = 0,1,\cdots,N \end{cases} \end{cases} \tag{3.32}$$

***Proof*:**

a- Obviously, for $i = 0,1,2,\cdots,k$, we have $b_{ij} = 0$, now for $i > k$ approximate $x^{i\alpha-\lambda}$ by terms of fractional-order Chebyshev series, we get

$$x^{i\alpha-\lambda} \cong \sum_{j=0}^{N} g_{ij} \cdot \bar{T}_i^\alpha(x) \tag{3.33}$$

by(3.23), we get

$$g_{ij} = \frac{4\cdot\alpha}{\pi\cdot c_i}\int_0^1 x^{i\alpha-\lambda} \cdot \bar{T}_i^\alpha(x) \cdot \frac{1}{x\cdot\sqrt{x^{-2\alpha}-1}}\, dx$$

$$= \frac{4\cdot\alpha}{\pi\cdot c_i}\int_0^1 x^{i\alpha-\lambda} \cdot \sum_{\ell=0}^{j} f_{j\ell} \cdot \frac{1}{x\cdot\sqrt{x^{-2\alpha}-1}}\, dx$$

where, $c_i = \begin{cases} 2 & i = 0 \\ 1 & i \geq 1 \end{cases}$

$$= \frac{4\cdot\alpha}{\pi\cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \int_0^1 x^{i\alpha-\lambda} \cdot x^{\ell\alpha} \cdot \frac{1}{x\cdot\sqrt{x^{-2\alpha}-1}}\, dx$$

$$= \frac{4\cdot\alpha}{\pi\cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \int_0^1 x^{\alpha(i+\ell)-\lambda-1} \cdot \frac{1}{\sqrt{x^{-2\alpha}-1}}\, dx$$

$$= \frac{4\cdot\alpha}{\pi\cdot c_i} \sum_{\ell=0}^{j} f_{j\ell} \int_0^\infty \left(\frac{u^2}{1+u^2}\right)^{\frac{1}{2\alpha}[\alpha(i+l)-\lambda-1]} \cdot u \frac{u^{\frac{1}{\alpha}-1}}{\alpha(1+u^2)^{1+\frac{1}{2\alpha}}}\, du$$

$$= \frac{4}{\pi\cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \cdot \int_0^\infty \frac{(u)^{(i+\ell)-\frac{\lambda}{\alpha}1}}{(1+u^2)^{\frac{1}{2}(i+\ell)-\lambda+1}}\, du$$

$$= \frac{2}{\pi\cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \int_0^\infty \frac{(t)^{\frac{1}{2}(i+\ell)-\frac{\lambda}{2\alpha}-\frac{1}{2}}}{(1+t)^{\frac{1}{2}(i+\ell)-\frac{\lambda}{2\alpha}+1}}\, dt$$

$$= \frac{2}{\pi\cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \,.\, B\left(\frac{1}{2}(i+\ell) - \frac{\lambda}{2\alpha} + \frac{1}{2}\ ,\frac{1}{2}\right) \tag{3.34}$$

$$= \frac{2}{\pi\cdot c_i} \cdot \sum_{\ell=0}^{j} f_{j\ell} \frac{\Gamma(\frac{1}{2}(i+\ell)-\frac{\lambda}{2\alpha}+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1}{2}(i+\ell)-\frac{\lambda}{2\alpha}+1\right)}$$

b- Obviously, for $i = 0,1,2,\cdots,k$, we have $h_{ij} = 0$, now for $i > k$ approximate $x^{i\alpha-\lambda}$ by $(N+1)$ terms of fractional-order Chebyshev series, we get

$$x^{i\alpha-\lambda} \cong \sum_{j=0}^{N} h_{ij} \cdot \bar{V}_i^\alpha(x) \tag{3.35}$$

by(3.24), we get

$$h_{ij} = \frac{2 \cdot \alpha}{\pi} \int_0^1 x^{i\alpha-\lambda} \cdot V_i^\alpha(x) \cdot x^{\alpha-1} \sqrt{(x^{-\alpha}-1)^{-1}}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \int_0^1 x^{i\alpha-\lambda} \cdot \sum_{\ell=0}^{j} f_{i\ell} \cdot x^{\alpha-1} \sqrt{(x^{-\alpha}-1)^{-1}}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^1 x^{i\alpha-\lambda} \cdot x^{l\alpha} \cdot x^{\alpha-1} \sqrt{(x^{-\alpha}-1)^{-1}}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^1 x^{\alpha(i+l+1)-\lambda-1} \cdot \sqrt{(x^{-\alpha}-1)^{-1}}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^1 G(x,i,\ell,\alpha,\lambda) \tag{3.36}$$

$$= \frac{2\cdot\alpha}{\pi}\ \frac{1}{\alpha} \sum_{\ell=0}^{j} f_{i\ell}\ \int_0^\infty \left[\left(\frac{u^2}{1+u^2}\right)^{\frac{1}{\alpha}}\right]^{\alpha(i+l+1)-\lambda-1} \cdot \frac{2u^2(1+u^2)-2u^4}{(1+u^2)^2} \cdot du$$

$$= \frac{4}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^\infty \frac{(u^2)^{(i+l+1)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+2}}{(1+u^2)^{(i+l+1)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+2}} \cdot du \tag{3.37}$$

$$= \frac{2}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^\infty \frac{(t)^{(i+l)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+\frac{5}{2}}}{(1+t)^{(i+l)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+3}} \cdot dt$$

$$= \frac{2}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \,.\, B\left((i+l)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+\frac{7}{2}\ ,-\frac{1}{2}\right)$$

$$= \frac{-4}{\sqrt{\pi}} \sum_{\ell=0}^{j} f_{i\ell} \cdot \frac{\Gamma\left((i+l)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+\frac{7}{2}\right)}{\Gamma\left((i+1)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+3\right)}$$

c- Obviously, for $i = 0,1,2,\cdots,k$, we have $r_{ij} = 0$, now for some $i > k$ approximate $x^{i\alpha-\lambda}$ by $(N+1)$ terms of fractional-order Chebyshev series, we get:

$$x^{i\alpha-\lambda} \cong \sum_{j=0}^{N} r_{ij} \cdot \bar{W}_i^\alpha(x) \tag{3.38}$$

by(3.25),we have

$$r_{ij} = \frac{2\cdot\alpha}{\pi} \int_0^1 x^{i\alpha-\lambda} \cdot \bar{W}_i^\alpha(x) \cdot x^{\alpha-1} \sqrt{x^{-\alpha}-1}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \int_0^1 x^{i\alpha-\lambda} \cdot \sum_{\ell=0}^{j} f_{i\ell} \cdot x^{\alpha-1} \sqrt{x^{-\alpha}-1}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^1 x^{i\alpha-\lambda} \cdot x^{l\alpha} \cdot x^{\alpha-1} \sqrt{x^{-\alpha}-1}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^1 x^{\alpha(i+l+1)-\lambda-1} \cdot \sqrt{x^{-\alpha}-1}\ dx$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} \int_0^1 G(x,i,\ell,\alpha,\lambda)\ dx \tag{3.39}$$

$$= \frac{2\cdot\alpha}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_\infty^0 \left[\left(\frac{1}{1+u^2}\right)^{\frac{1}{\alpha}}\right]^{\alpha(i+l+1)-\lambda-1} \cdot u \cdot \frac{1}{\alpha}\left(\frac{1}{1+u^2}\right)^{\frac{1}{\alpha}-1} \cdot \frac{-2u}{(1+u^2)^2} du$$

$$= \frac{-4}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^\infty \left(\frac{1}{1+u^2}\right)^{(i+l+1)-\frac{\lambda}{\alpha}-\frac{1}{\alpha}+\frac{1}{\alpha}-1} \cdot \frac{u^2}{(1+u^2)^2} \cdot du$$

$$= \frac{-4}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^\infty \frac{u^2}{(1+u^2)^{i+l-\frac{\lambda}{\alpha}+2}} \cdot du \tag{3.40}$$

$$= \frac{-4}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^\infty \frac{t}{(1+t)^{i+l-\frac{\lambda}{\alpha}+2}} \cdot \frac{1}{2}\ t^{-\frac{1}{2}}\, dt$$

$$= \frac{-2}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \int_0^\infty \frac{t^{\frac{1}{2}}}{(1+t)^{i+l-\frac{\lambda}{\alpha}+2}} \cdot dt \tag{3.41}$$

$$= \frac{-2}{\pi} \cdot \sum_{\ell=0}^{j} f_{i\ell} \cdot B\left(\frac{3}{2}\ , i + l - \frac{\lambda}{\alpha} + \frac{1}{2}\right)$$

$$= \frac{-1}{\sqrt{\pi}} \cdot \sum_{\ell=0}^{j} f_{i\ell} \cdot \frac{\Gamma\left(i+l-\frac{\lambda}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(i+l-\frac{\lambda}{\alpha}+2\right)}\ .$$

Theorem (3.3.14) :

Let $\bar{T}_\alpha(x)$, $\bar{V}_\alpha(x)$ and $\bar{W}_\alpha(x)$ be fractional shifted chebyshev vectors respectively, D^λ is the $(n+1)(n+1)$operational matrix of fractional derivative of order $\lambda > 0$ in Caputo sense and $\alpha \in N_0$ or $\alpha > \lceil \lambda \rceil$ when $\alpha \notin N$ then:

a- $D^\lambda = F^{(1)}\bar{\Delta}^\lambda G\ \bar{T}_\alpha(x)$ for fractional order shifted Chebyshev polynomial of first kind (3.42)

b- $D^\lambda = F^{(3)}\bar{\Delta}^\lambda H\ \bar{V}_\alpha(x)$ for fractional order shifted Chebyshev polynomial of third kind (3.43)

c- $D^\lambda = F^{(4)}\bar{\Delta}^\lambda R\ \bar{W}_\alpha(x)$ for fractional order shifted Chebyshev polynomial of fourth kind (3.44)

where, $G = (g_{ij})$, $H = (h_{ij})\ and\ \ R = (r_{ij})$, are given in theorem(3.3.13), and $\bar{\Delta}^\lambda$ is given in equation(3.29b)

Proof:

a- Application remark(1.4.4)(i), we can write λth order fractional derivative of $\bar{T}_\alpha(x)$ as

$$D^\lambda \bar{T}_\alpha(x) = F^{(1)} D^\lambda X_\alpha(x) = F^{(1)} \bar{\Delta}^\lambda X_\alpha^\lambda(x)$$
$$\cong\ F^{(1)} \bar{D}_\lambda \cdot G.\bar{T}_\alpha(x) = D^{(\lambda)} \cdot \bar{T}_\alpha(x)$$

b, c have the same prove in **(a)**.

Example (3.3.10):

Consider the following multi-fractional order nonlinear differential equation,

$$D^3\ u(x) + D^{\frac{5}{2}}\ u(x) + u^2(x) = x^4 \tag{3.45}$$

with mixed boundary conditions,

$$u(0) = 0\ ,\ u^{(1)}(1) = 2\ ,\ \ u^{(2)}(1) = 2 \tag{3.46}$$

To find the approximate solution with $m = 3$, $\alpha = 2$ the order of $\overline{T_\alpha}(x)$ polynomial, such that exact solution is $y(x) = x^2$.

By using equation(3.30) and lemma(3.3.3)with equation(3.42),we get

$$D^{\frac{5}{2}} \cong F^{(1)}.\bar{\Delta}^{\frac{5}{2}} \cdot G.\bar{T}_2(x) \tag{3.47}$$

also, from(3.23), we have

$\overline{T_\alpha}(x) = F^{(1)} X_2$

$$\begin{bmatrix} 1 \\ 2x^{\alpha}-1 \\ 8x^{2\alpha}-8x^{\alpha}+1 \\ 32x^{3\alpha}-48x^{2\alpha}+18x^{\alpha}-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 8 & 0 \\ -1 & 18 & -48 & 32 \end{bmatrix} \begin{bmatrix} 1 \\ x^2 \\ x^4 \\ x^6 \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 8 & 0 \\ -1 & 18 & -48 & 32 \end{bmatrix} \qquad (3.48a)$$

from(3.30), we have

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.383 & 0.721 & 0.0088 & -0.01 \\ 1.052 & 0.708 & 0.182 & -0.0017 \end{bmatrix} \qquad (3.48b)$$

from(3.29b), N=3, k=1 , $\alpha = 2$ and $\lambda = \frac{5}{2}$, we have

$$\overline{\Delta}^{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 18.054 & 0 \\ 0 & 0 & 0 & 61.9 \end{bmatrix}$$

(3.48c)

by substituting(3.48)(a),(b)and(c) in (3.47), we get

$D^{\frac{5}{2}} = F^{(1)}.\overline{\Delta}^{\frac{5}{2}} \cdot G.\overline{T_2}(x) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 199.7501 & 351.6932 & 478.7938 & 590.2957 \\ 8652888 & 1716.7250 & 2483.2904 & 3190.9273 \end{bmatrix} . \begin{bmatrix} 1 \\ 2x^2-1 \\ 8x^4-8x^2+1 \\ 32x^6-48x^4+18x^2-1 \end{bmatrix} \qquad (3.49)$$

by using the first root of $x^{\alpha} = \frac{1}{2}$ of the polynomial $T^{\alpha}_{m+1-\lambda}(x)$, substituting this root in(3.45), we get

$$-143.2792\, C_2 - 1862.4595\, C_3 + (C_0 - C_2)^2 = 0.25 \qquad (3.50)$$

from(3.46), we obtain

$$u(0) = C_0 - C_1 + C_2 - C_3 = 0 \qquad (3.51a)$$

$$u^{(1)}(1) = 4\, C_1 + 16\, C_2 + 36 C_3 = 2 \qquad (3.51b)$$

$$u^{(2)}(1) = 4C_1 - 80\, C_2 - 444 C_3 = 2 \qquad (3.51c)$$

from(3.50),(3.51)(a),(b)and(c), we obtain

$C_0 = 0.5000$, $C_1 = 0.5000$, $C_2 = C_3 = 0$.

Then, the approximate solution is

$$y(x) = 0.5000 + 0.5000(2x^2-1) + 0$$
$$= 0.5000 + x^2 - 0.5000 = x^2$$

Example(3.3.11):

Consider the following multi-fractional order nonlinear differential equation,

$$D^4u(x)+D^{\frac{7}{2}}u(x)+u^{(3)}(x)=x^9 \quad (3.52)$$

with mixed boundary conditions,

$$u(0)=0\ ,\ u^{(1)}(1)=3\ ,\ u^{(2)}(1)=6\ ,\ \ u^{(3)}(1)=6 \quad (3.53)$$

To find the approximate solution with $m=4$, $\alpha=3$ the order of fractional shifted Chebyshev polynomial of third kind such that the exact solution is $y(x)=x^3$.

By using equation(3.31) and lemma(3.3.3)with equation(3.43), we have

$$D^{\frac{7}{2}} \cong F^{(3)}.\bar{\Delta}^{\frac{7}{2}}\cdot H.\overline{V_3}(x) \quad (3.54)$$

also, from(3.24), we get

$$\overline{V_\alpha}(x)=F^{(3)}X_3$$

$$\begin{bmatrix} 1 \\ 4\,x^\alpha-3 \\ 16\,x^{2\alpha}-20x^\alpha+5 \\ 64\ x^{3\alpha}-112\ x^{2\alpha}+56x^\alpha-7 \\ 256\ x^{4\alpha}-576\ x^{3\alpha}+432\ x^{2\alpha}-120x^\alpha+9 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 \\ 9 & -120 & 432 & -576 & 256 \end{bmatrix}\begin{bmatrix} 1 \\ x^3 \\ x^6 \\ x^9 \\ x^{12} \end{bmatrix}$$

$$F=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 \\ 9 & -120 & 432 & -576 & 256 \end{bmatrix} \quad (3.55a)$$

by(3.31), we obtain

$$H=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4.074 & -6.403 & -10.024 & -13.999 & -18 \\ -4.656 & -6.726 & -10.112 & -14.005 & -18 \\ -5.174 & -7.055 & -10.239 & -14.031 & -18.001 \end{bmatrix} \quad (3.55b)$$

from(3.29b), N=4, k=1 , $\alpha=3$, and $\lambda=\frac{7}{2}$, we get

$$\bar{\Delta}^{\frac{7}{2}}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma((k+2)\alpha+1)}{\Gamma((k+2)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 770.5041 & 0 & 0 \\ 0 & 0 & 0 & 1261 & 0 \\ 0 & 0 & 0 & 0 & 4015 \end{bmatrix}$$

(3.55c)

by substituting(3.55)(a),(b)and(c)in(3.54), we get

$$D^{\frac{7}{2}} \cong F^{(3)}.\bar{\Delta}^{\frac{7}{2}}\cdot H.\overline{V_3}(x)=$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -50224.5392 & -107636.3407 & -171421.7521 & -241001.3544 & -315931.3371 \\ -24186.0492 & -39865.9347 & -48861.4308 & -52403.8310 & -51097.9840 \\ -3.2922\times10^6 & -6.8854\times10^6 & -1.0755\times10^7 & -1.4883\times10^7 & -1.9253\times10^7 \end{bmatrix}.\begin{bmatrix} 1 \\ 4\,x^3-3 \\ 16\,x^6-20x^3+5 \\ 64\ x^9-112\ x^6+56x^3-7 \\ 256\ x^{12}-576\ x^9+432\ x^6-120x^3+9 \end{bmatrix}$$

(3.56)

from using the first root of $x^\alpha=\frac{3}{4}$ of the polynomial $V_{m+1-\lambda}^{\alpha}(x)$, substituting this root in(3.52), we get

$125956.5914c_2 + 163874.1185c_3 + 23052154.51c_4 +$
$(c_0 - c_2 - c_3)^3$=0.421875 (3.57)

from(3.53), we obtain

$c_0 - 3\, c_1 + 5\, c_2 - 7\, c_3 + 9\, c_4 = 0$ (3.58a)

$12\, c_1 + 36\, c_2 + 72\, c_3 + 120c_4 = 3$ (3.58b)

$24\, c_1 + 360\, c_2 + 1584\, c_3 + 4560\, c_4 = 6$ (3.58c)

$24\, c_1 + 1800\, c_2 + 19152\, c_3 + 98736\, c_4 = 6$ (3.58d)

by(3.57),(3.58)(a),(b),(c)and(d), we get

$c_0 = 0.7500\,, c_1 = 0.2500\ \ , c_2 = c_3 = c_4 = 0$.

Then, the approximate solution is

$y(x) = 0.7500 + 0.2500(4\,x^3 - 3) + 0$
$\quad = 0.7500 + x^3 - 0.7500$
$\quad = x^3$.

Table (3.2)

x	Approximate solution with $\alpha = 1.5$	Approximate solution with $\alpha = 2.5$	Approximate solution with $\alpha = 3$	Exact solution
0.1	-0.0002452	0.013	0.001	0.001
0.2	0.002415	0.032	0.008	0.008
0.3	0.014	0.073	0.027	0.027
0.4	0.04	0.138	0.064	0.064
0.5	0.086	0.232	0.125	0.125
0.6	0.159	0.354	0.216	0.216
0.7	0.266	0.507	0.343	0.343
0.8	0.414	0.692	0.512	0.512
0.9	0.609	0.913	0.729	0.729

Example(3.3.12):

Consider the following multi-fractional order nonlinear differential equation,

$D^5 u(x) + D^{\frac{9}{2}} u(x) + u^4(x) = x^{16}$ (3.59)

with mixed boundary condition,

$u(0) = 0\ ,\ u^{(1)}(1) = 4\ ,\ u^{(2)}(1) = 12\ ,\ \ u^{(3)}(1) = 24,\ u^{(4)}(0) = 24$

(3.60)

To find the approximate solution with $m = 5\ ,\ \alpha = 4$ the order of fractional shifted Chebyshev of fourth kind such that the exact solution is $(x) = x^4$.

By using equation (3.32)and lemma(3.3.3)with equation(3.44), we obtain

$$D^{\frac{9}{2}} \cong F^{(4)}.\bar{\Delta}^{\frac{9}{2}} \cdot R.\overline{W_4}(x) \tag{3.61}$$

by(3.25), we obtain

$$\overline{W_\alpha}(x) = F^{(4)}X_4$$

$$\begin{bmatrix} 1 \\ 2x^{\alpha}+1 \\ 8x^{2\alpha}-3 \\ 32x^{3\alpha}-16x^{2\alpha}-14x^{\alpha}+5 \\ 128x^{4\alpha}-128x^{3\alpha}-32x^{2\alpha}+48x^{\alpha}-7 \\ 512x^{5\alpha}-768x^{4\alpha}+96x^{3\alpha}+272x^{2\alpha}-110x^{\alpha}+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -3 & 0 & 8 & 0 & 0 & 0 \\ 5 & -14 & -16 & 32 & 0 & 0 \\ -7 & 48 & -32 & -128 & 128 & 0 \\ 9 & -110 & 272 & 96 & -768 & 512 \end{bmatrix} \begin{bmatrix} 1 \\ x^4 \\ x^8 \\ x^{12} \\ x^{16} \\ x^{20} \end{bmatrix}$$

$$F^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -3 & 0 & 8 & 0 & 0 & 0 \\ 5 & -14 & -16 & 32 & 0 & 0 \\ -7 & 48 & -32 & -128 & 128 & 0 \\ 9 & -110 & 272 & 96 & -768 & 512 \end{bmatrix} \tag{3.62a}$$

from(3.32), we have

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.281 & -0.549 & 0.184 & -0.03 & 0.016 & -0.0098 \\ -0.134 & -0.299 & -0.053 & 0.035 & -0.0022 & 0.00070 \\ -0.082 & -0.196 & -0.092 & 0.0035 & 0.0077 & -0.0002 \\ -0.057 & -0.142 & -0.094 & -0.019 & 0.0046 & 0.0018 \end{bmatrix}$$

(3.62b)

from(3.29b), N=5 , k=1 , $\alpha = 4$, and $\lambda = \frac{9}{2}$, we get

$$\bar{\Delta}^{\frac{9}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma((k+2)\alpha+1)}{\Gamma((k+2)\alpha+1-\lambda)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma((k+3)\alpha+1)}{\Gamma((k+3)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.44\times 10^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.413\times 10^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.529\times 10^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4688\times 10^5 \end{bmatrix} \tag{3.62c}$$

by substituting(3.62)(a),(b)and(c)in(3.61), we obtain

$$D^{\frac{9}{2}} \cong F^{(4)}.\bar{\Delta}^{\frac{9}{2}} \cdot R.\overline{W_4}(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -7.792\times 10^3 & -1.522\times 10^4 & 5.102\times 10^3 & -831.84 & 443.648 & -271.734 \\ -1.308\times 10^5 & -2.961\times 10^5 & -6.809\times 10^4 & 3.989\times 10^4 & -3.29\times 10^3 & 1.308\times 10^3 \\ -9.883\times 10^5 & -2.469\times 10^6 & -1.589\times 10^6 & -8.108\times 10^4 & 1.585\times 10^5 & -5.885\times 10^3 \\ -4.756\times 10^6 & -1.257\times 10^7 & -1.176\times 10^7 & -4.885\times 10^6 & 2.078\times 10^5 & 4.486\times 10^5 \end{bmatrix}$$

$$\cdot\begin{bmatrix} 1 \\ 2x^4+1 \\ 8x^8-3 \\ 32x^{12}-16x^8-14x^4+5 \\ 128x^{16}-128x^{12}-32x^8+48x^4-7 \\ 512x^{20}-768x^{16}+96x^{12}+272x^8-110x^4+9 \end{bmatrix} \quad (3.63)$$

from using the first root of $x^{\alpha} = \frac{1}{4}$ of the polynomial $W^{\alpha}_{m+1-\lambda}(x)$, substituting this root in(3.59), we obtain

$$-9793.1183\, c_2 + 105638.4983\, c_3 + 2210690\, c_4 + 7550000\, c_5 + (c_0 + 2c_1 - c_2 - 2c_3 + c_4 + 2c_5)^4 = 0.0625 \quad (3.64)$$

by(3.60), we get

$$c_0 + c_1 - 3\,c_2 + 5\,c_3 - 7\,c_4 + 9\,c_5 = 0 \quad (3.65a)$$
$$8\,c_1 + 64\,c_2 + 200\,c_3 + 448\,c_4 + 840\,c_5 = 4 \quad (3.65b)$$
$$24\,c_1 + 448\,c_2 + 3160\,c_3 + 12608\,c_4 + 36824\,c_5 = 12 \quad (3.65c)$$
$$48c_1 + 2688c_2 + 36528c_3 + 251520c_4 + 1137072c_5 = 24 \quad (3.65d)$$
$$48\,c_1 + 13440\,c_2 - 336\,c_3 + 1152\,c_4 - 2640\,c_5 = 24 \quad (3.65e)$$

from(3.64),(3.65)(a),(b),(c),(d)and(e), we have

$c_0 = -0.5000$, $c_1 = 0.5000$, $c_2 = c_3 = c_4 = c_5 = 0$

Then, the approximate solution is

$$y(x) = -0.5000 + 0.5000(2\,x^4 + 1) + 0$$
$$= -0.5000 + x^4 + 0.5000 = x^4$$

The following proposition is proved by fractional shifted Chebyshev polynomials of first, third and fourth kinds as well as in fractional shifted Chebyshev polynomial of second kind, see[33] .

Proposition(3.3.1):

The operational matrix of fractional derivative α, can be computed as,

$$D^{\alpha} = F^{(i)}\bar{\Delta}^{\alpha}F^{(i)^{-1}}\bar{\emptyset}_{\alpha}(x) \quad \text{where, } i=1,3,4 \quad (3.66)$$

where, $\bar{\emptyset}_{\alpha}(x) = \{\bar{T}_{\alpha}(x), \bar{V}_{\alpha}(x), \bar{W}_{\alpha}(x)\}$

also, $\bar{\Delta}^{\alpha}$ is a $(n+1)(n+1)$ operational matrix of fractional derivative.

$$\bar{\Delta}^{\alpha} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \ddots & \vdots & \vdots \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} & 0 \end{bmatrix} \quad (3.67)$$

Example (3.3.13):

Consider the following multi-fractional order nonlinear differential equation,

$$D^3\, u(x) + D^{\frac{5}{2}}\, u(x) + u^2(x) = x^{2\left(\frac{5}{2}\right)} + \frac{15\sqrt{\pi}}{8} \quad (3.68)$$

with mixed conditions,

$$u(0) = 0\ ,\ u^{(1)}(1) = \frac{5}{2}\ ,\ u^{(2)}(1) = \frac{15}{4} \qquad (3.69)$$

To solve the problem with $m = 3$ and $\alpha = \frac{5}{2}$ the order of fractional shifted Chebyshev polynomial of first kind with the exact solution $y(x) = x^{\frac{5}{2}}$.

By using equation(3.66), we get

$$D^{\frac{5}{2}} = F^{(1)}\bar{\Delta}^{\frac{5}{2}} \cdot {F^{(1)}}^{-1}.\bar{T}_{\frac{5}{2}}(x) \qquad (3.70)$$

by substituting N=3 , $\alpha = \frac{5}{2}$ in equation(3.67)

$$\bar{\Delta}^{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3.323 & 0 & 0 & 0 \\ 0 & 36.108 & 0 & 0 \\ 0 & 0 & 116.953 & 0 \end{bmatrix}$$

(3.71a)

from(3.23), we have

$$\overline{T_\alpha}(x) = F^{(1)}X_{\frac{5}{2}}$$

$$\begin{bmatrix} 1 \\ 2x^{\alpha} - 1 \\ 8x^{2(\alpha)} - 8x^{\alpha} + 1 \\ 32x^{3(\alpha)} - 48x^{2(\alpha)} + 18x^{\alpha} - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 8 & 0 \\ -1 & 18 & -48 & 32 \end{bmatrix} \begin{bmatrix} x \\ x^{\frac{5}{2}} \\ x^{2\left(\frac{5}{2}\right)} \\ x^{3\left(\frac{5}{2}\right)} \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 8 & 0 \\ -1 & 18 & -48 & 32 \end{bmatrix} \qquad (3.71b)$$

$${F^{(1)}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.375 & 0.5 & 0.125 & 0 \\ 0.313 & 0.469 & 0.188 & 0.031 \end{bmatrix} \qquad (3.71c)$$

by substituting(3.71)(a),(b)and(c)in(3.70), we obtain

$$D^{\frac{5}{2}} = F^{(1)}\Delta^{\frac{5}{2}} \cdot {F^{(1)}}^{-1}.\bar{T}_{\frac{5}{2}}(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 6.647 & 0 & 0 & 0 \\ 117.846 & 144.433 & 0 & 0 \\ 596.666 & 1005 & 467.814 & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 2x^{\frac{5}{2}} - 1 \\ 8x^{2\left(\frac{5}{2}\right)} - 8x^{\frac{5}{2}} + 1 \\ 32x^{3\left(\frac{5}{2}\right)} - 48x^{2\left(\frac{5}{2}\right)} + 18x^{\frac{5}{2}} - 1 \end{bmatrix} \qquad (3.72)$$

from using the first root of $x^{\frac{5}{2}} = \frac{1}{2}$ of the polynomial $T^{\frac{5}{2}}{}_{m+1-\alpha}(x)$, substituting this root in(3.68), we get

$$6.647C_1 + 393.4912C_2 - 255.1413C_3 + (C_0 - C_2)^2 = 3.5733 \qquad (3.73)$$

by(3.69), we have

$$u(0) = C_0 - C_1 + C_2 - C_3 = 0 \qquad (3.74a)$$

$$u^{(1)}(1) = 5\,C_1 + 20\,C_2 - 99\,C_3 = \frac{5}{2} \qquad (3.74b)$$

$$u^{(2)}(1) = \frac{15}{2}C_1 + 130\,C_2 + \frac{183}{2}C_3 = \frac{15}{4} \qquad (3.74c)$$

from(3.73),(3.74)(a),(b)and(c), we get

$C_0 = C_1 = 0.5000,\;\; C_2 = C_3 = 0$.

Then, the approximate solution is

$$y(x) = 0.5000 + 0.5000\left(2x^{\frac{5}{2}} - 1\right) + 0 = x^{\frac{5}{2}}$$

Example(3.3.14):

Consider the following multi-fractional order nonlinear differential equation,

$$D^4 u(x) + D^{\frac{7}{2}} u(x) + u^{(3)}(x) = \left(x^{\frac{7}{2}}\right)^3 + 11.632 \qquad (3.75)$$

with mixed boundary condition ,

$$u(0) = 0\;,\; u^{(1)}(1) = \frac{7}{2}\;,\; u^{(2)}(1) = \frac{35}{4}\;,\; u^{(3)}(1) = \frac{105}{8} \qquad (3.76)$$

To find the approximate solution with $m = 3$,$\alpha = \frac{7}{2}$ the order of fractional shifted Chebyshev polynomial of third kind with the exact solution $y(x) = x^{\frac{7}{2}}$.

By using equation(3.66), we have

$$D^{\frac{7}{2}} = F^{(3)} \bar{\Delta}^{\frac{7}{2}} \cdot F^{(3)^{-1}} . \bar{V}_{\frac{7}{2}}(x) \qquad (3.77)$$

from substituting N=4 , $\alpha = \frac{7}{2}$ in equation(3.67)

$$\bar{\Delta}^{\frac{7}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 11.632 & 0 & 0 & 0 & 0 \\ 0 & 433.298 & 0 & 0 & 0 \\ 0 & 0 & 2361 & 0 & 0 \\ 0 & 0 & 0 & 7326 & 0 \end{bmatrix} \qquad (3.78a)$$

by(3.24), we have

$$\bar{V}_\alpha(x) = F^{(3)} X_{\frac{7}{2}}$$

$$\begin{bmatrix} 1 \\ 4\,x^{\alpha} - 3 \\ 16\,x^{2(\alpha)} - 20\,x^{\alpha} + 5 \\ 64\,x^{3(\alpha)} - 112\,x^{2(\alpha)} + 56\,x^{\alpha} - 7 \\ 256\,x^{4(\alpha)} - 576\,x^{3(\alpha)} + 432\,x^{2(\alpha)} - 120\,x^{\alpha} + 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 \\ 9 & -120 & 432 & -576 & 256 \end{bmatrix} \begin{bmatrix} 1 \\ x^{\frac{7}{2}} \\ x^{2\left(\frac{7}{2}\right)} \\ x^{3\left(\frac{7}{2}\right)} \\ x^{4\left(\frac{7}{2}\right)} \end{bmatrix}$$

$$F^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 \\ 9 & -120 & 432 & -576 & 256 \end{bmatrix} \qquad (3.78b)$$

$$F^{(3)^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0.25 & 0 & 0 & 0 \\ 0.625 & 0.313 & 0.063 & 0 & 0 \\ 0.547 & 0.328 & 0.109 & 0.016 & 0 \\ 0.492 & 0.328 & 0.141 & 0.035 & 0.003906 \end{bmatrix} \quad (3.78c)$$

from substituting(3.78)(a),(b)and(c)in(3.77), we obtain

$$D^{\frac{7}{2}} = F^{(3)} \bar{\Delta}^{\frac{7}{2}} \cdot F^{(3)^{-1}} . \bar{V}_{\frac{7}{2}}(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 46.527 & 0 & 0 & 0 & 0 \\ 4.967 \times 10^3 & 1.733 \times 10^3 & 0 & 0 & 0 \\ 5.869 \times 10^4 & 3.509 \times 10^4 & 9.444 \times 10^3 & 0 & 0 \\ 3.147 \times 10^5 & 2.372 \times 10^5 & 1.201 \times 10^5 & 2.931 \times 10^4 & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 4\,x^{\frac{7}{2}} - 3 \\ 16\,x^{2\left(\frac{7}{2}\right)} - 20\,x^{\frac{7}{2}} + 5 \\ 64\,x^{3\left(\frac{7}{2}\right)} - 112\,x^{2\left(\frac{7}{2}\right)} + 56\,x^{\frac{7}{2}} - 7 \\ 256\,x^{4\left(\frac{7}{2}\right)} - 576\,x^{3\left(\frac{7}{2}\right)} + 432\,x^{2\left(\frac{7}{2}\right)} - 120\,x^{\frac{7}{2}} + 9 \end{bmatrix} \quad (3.79)$$

by using the first root of $x^{\frac{7}{2}} = \frac{3}{4}$ of the polynomial $V^{\frac{7}{2}}{}_{m+1-\alpha}(x)$, substituting this root in(3.75), we obtain

$$46.527\,c_1 + 15466.7306\,c_2 + 214166.8205\,c_3 + 1003744.909\,c_4 + (c_0 - c_2 - c_3)^3 = 12.0538 \quad (3.80)$$

from(3.76), we get

$$u(0) = c_0 - 3\,c_1 + 5\,c_2 - 7\,c_3 + 9\,c_4 = 0 \quad (3.81a)$$

$$u^{(1)}(1) = 14\,c_1 + 42\,c_2 + 84\,c_3 + 140\,c_4 = \frac{7}{2} \quad (3.81b)$$

$$u^{(2)}(1) = 35\,c_1 + 497\,c_2 + 2170\,c_3 + 6230\,c_4 = \frac{35}{4} \quad (3.81c)$$

$$u^{(3)}(1) = 52.5\,c_1 + 3097.5\,c_2 + 31479\,c_3 + 599412\,c_4 = \frac{105}{8} \quad (3.81d)$$

by(3.80),(3.81)(a),(b),(c)and(d), we obtain

$c_0 = 0.7500\,, c_1 = 0.2500\,, c_2 = c_3 = c_4 = 0$.

Then, the approximate solution is

$$y(x) = 0.7500 + 0.2500\left(4\,x^{\frac{7}{2}} - 3\right) + 0 = x^{\frac{7}{2}}$$

Example(3.3.15):

Consider the following multi-fractional order nonlinear differential equation,

$$D^5 u(x) + \mathrm{D}^{\frac{9}{2}} u(x) + u^4(x) = \left(x^{\frac{9}{2}}\right)^4 + 52.343 \quad (3.82)$$

with mixed boundary condition,

$$u(0) = 0 \;,\; u^{(1)}(1) = \frac{9}{2} \;,\; u^{(2)}(1) = \frac{63}{4} \;,\quad u^{(3)}(1) = \frac{315}{8},\; u^{(4)}(0) = 0 \quad (3.83)$$

For solving the problem with $m = 5$,and $\alpha = \frac{9}{2}$ the order of fractional shifted Chebshev polynomial of fourth kind with the exact solution $\mathrm{y}(x) = x^{\frac{9}{2}}$.

From using equation(3.66), we have

$$D^{\frac{9}{2}} = F^{(4)}\bar{\Delta}^{\frac{9}{2}} \cdot F^{(4)^{-1}}.\bar{W}_{\frac{9}{2}}(x) \qquad (3.84)$$

by substituting N=5 , $\alpha = \frac{9}{2}$ in equation(3.67)

$$\Delta^{\frac{9}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)} & \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 52.343 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6.933\times10^{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.364\times10^{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.773\times10^{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.374\times10^{5} & 0 \end{bmatrix}$$

(3.85a)

from(3.25), we have

$$\overline{W_\alpha}(x) = F^{(4)}X_{\frac{9}{2}}$$

$$\begin{bmatrix} 1 \\ 2x^{\frac{9}{2}}+1 \\ 8x^{2\left(\frac{9}{2}\right)}-3 \\ 32x^{3\left(\frac{9}{2}\right)}-16x^{2\left(\frac{9}{2}\right)}-14x^{\frac{9}{2}}+5 \\ 128x^{4\left(\frac{9}{2}\right)}-128x^{3\left(\frac{9}{2}\right)}-32x^{2\left(\frac{9}{2}\right)}+48x^{\frac{9}{2}}-7 \\ 512x^{5\left(\frac{9}{2}\right)}-768x^{4\left(\frac{9}{2}\right)}+96x^{3\left(\frac{9}{2}\right)}+272x^{2\left(\frac{9}{2}\right)}-110x^{\frac{9}{2}}+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -3 & 0 & 8 & 0 & 0 & 0 \\ 5 & -14 & -16 & 32 & 0 & 0 \\ -7 & 48 & -32 & -128 & 128 & 0 \\ 9 & -110 & 272 & 96 & -768 & 512 \end{bmatrix} \begin{bmatrix} 1 \\ x^{\frac{9}{2}} \\ x^{2\left(\frac{9}{2}\right)} \\ x^{3\left(\frac{9}{2}\right)} \\ x^{4\left(\frac{9}{2}\right)} \\ x^{5\left(\frac{9}{2}\right)} \end{bmatrix}$$

$$F^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -3 & 0 & 8 & 0 & 0 & 0 \\ 5 & -14 & -16 & 32 & 0 & 0 \\ -7 & 48 & -32 & -128 & 128 & 0 \\ 9 & -110 & 272 & 96 & -768 & 512 \end{bmatrix} \qquad (3.85b)$$

$$F^{(4)^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.375 & 0 & 0.125 & 0 & 0 & 0 \\ -0.188 & 0.219 & 0.063 & 0.031 & 0 & 0 \\ 0.148 & 0.031 & 0.094 & 0.031 & 7.813\times10^{-3} & 0 \\ -0.066 & 0.113 & 0.063 & 0.041 & 0.012 & 1.953\times10^{-3} \end{bmatrix} \qquad (3.85c)$$

by substituting(3.85)(a),(b)and(c)in(3.84), we get

$$D^{\frac{9}{2}} = F^{(4)}\bar{\Delta}^{\frac{9}{2}} \cdot F^{(4)^{-1}}.\bar{W}_{\frac{9}{2}}(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 104.686 & 0 & 0 & 0 & 0 & 0 \\ -2.773\times10^{4} & 2.773\times10^{4} & 0 & 0 & 0 & 0 \\ 8.184\times10^{5} & -5.546\times10^{4} & 2.545\times10^{5} & 0 & 0 & 0 \\ -9.595\times10^{6} & 7.652\times10^{6} & 1.2\times10^{6} & 1.109\times10^{6} & 0 & 0 \\ 1.049\times10^{8} & -3.224\times10^{7} & 2.765\times10^{7} & 6.744\times10^{6} & 3.35\times10^{6} & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 2x^{\frac{9}{2}}+1 \\ 8x^{2\left(\frac{9}{2}\right)}-3 \\ 32x^{3\left(\frac{9}{2}\right)}-16x^{2\left(\frac{9}{2}\right)}-14x^{\frac{9}{2}}+5 \\ 128x^{4\left(\frac{9}{2}\right)}-128x^{3\left(\frac{9}{2}\right)}-32x^{2\left(\frac{9}{2}\right)}+48x^{\frac{9}{2}}-7 \\ 512x^{5\left(\frac{9}{2}\right)}-768x^{4\left(\frac{9}{2}\right)}+96x^{3\left(\frac{9}{2}\right)}+272x^{2\left(\frac{9}{2}\right)}-110x^{\frac{9}{2}}+9 \end{bmatrix}$$

(3.86)

from using the first root of $x^{\frac{9}{2}} = \frac{1}{2}$ of the polynomial $W^{\frac{9}{2}}{}_{m+1-\alpha}(x)$, substituting this root in equation(3.82), we get

$$104.686\, c_1 + 92977.6787\, c_2 + 1990600\, c_3 + 13059000\, c_4 + 96280000\, c_5 + (c_0 + 2c_1 - c_2 - 2c_3 + c_4 + 2c_5)^4 = 52.4055 \quad (3.87)$$

by(3.83), we have

$$c_0 + c_1 - 3\, c_2 + 5\, c_3 - 7\, c_4 + 9\, c_5 = 0 \quad (3.88a)$$

$$9\, c_1 + 72\, c_2 + 225\, c_3 + 504\, c_4 + 12465\, c_5 = \frac{9}{2} \quad (3.88b)$$

$$31.5\, c_1 + 576\, c_2 + 4027.5\, c_3 + 16020\, c_4 + 505911.5\, c_5 = \frac{63}{4} \quad (3.88c)$$

$$78.75\, c_1 + 4032\, c_2 + 53484.75\, c_3 + 364050\, c_4 + 6713808.75\, c_5 = \frac{315}{8} \quad (3.88d)$$

$$24192\, c_2 - 48384\, c_3 + 9303552\, c_4 - 55579392\, c_5 = 0 \quad (3.88e)$$

from(3.87),(3.88)(a),(b),(c),(d),(e), we get

$c_0 = -0.5000$, $c_1 = 0.5000$, $c_2 = c_3 = c_4 = c_5 = 0$.

Then, the approximate solution is

$$y(x) = -0.5000 + 0.5000\left(2\, x^{\frac{9}{2}} + 1\right) + 0$$

$$= -0.5000 + x^{\frac{9}{2}} + 0.5000 = x^{\frac{9}{2}}$$

3.4 The Couple Fractional Order for Shifted Chebyshev polynomials:

In this section the multi-fractional order nonlinear differential equation has been solved with two different fractional order shifted Chebyshev polynomials such that one of then make as axillary fractional order two other and the following examples give the complete idea of the approximation.

Example(3.3.16):

Consider the following multi-fractional order nonlinear differential equation,

$$D^4 u(x) + D^{\frac{7}{2}} u(x) + D^{\frac{5}{2}} u(x) + u^{(3)}(x) = 6.7720\, x^{\frac{1}{2}} + x^9 \quad (3.89)$$

with mixed boundary condition,

$$u(0) = 0\, ,\ u^{(1)}(1) = 3\ ,\ u^{(2)}(1) = 6\, , u^{(3)}(1) = 6 \quad (3.90)$$

To solve the approximate solution with $m = 4,$ and $\alpha_1 = 3$, $\alpha_2 = 2$ the couple order for shifted Chebyshev polynomial of first kind such that the exact solution is $y(x) = x^3$.

By using equation(3.30) and lemma(3.3.3) and equation(3.42), we get

$$D^{\frac{7}{2}} \cong F^{(1)}.\bar{\Delta}^{\frac{7}{2}} \cdot G.\bar{T}_3(x) \quad (3.91)$$

also, from(3.23), we have

$$\overline{T_\alpha}(x) = F^{(1)}X_3$$

$$\begin{bmatrix} 1 \\ 2\,x^3-1 \\ 8\,x^6-8\,x^3+1 \\ 32\,x^9-48\,x^6+18\,x^3-1 \\ 128\,x^{12}-256\,x^9+160\,x^6-32\,x^3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -8 & 8 & 0 & 0 \\ -1 & 18 & -48 & 32 & 0 \\ 1 & -32 & 160 & -256 & 128 \end{bmatrix} \begin{bmatrix} 1 \\ x^3 \\ x^6 \\ x^9 \\ x^{12} \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -8 & 8 & 0 & 0 \\ -1 & 18 & -48 & 32 & 0 \\ 1 & -32 & 160 & -256 & 128 \end{bmatrix} \tag{3.92a}$$

from(3.30). we get

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.344 & 0.724 & 0.028 & -0.014 & 0.0052 \\ 1.034 & 0.705 & 0.192 & 0.0016 & -0.0013 \\ 0.869 & 0.659 & 0.272 & 0.048 & -0.00013 \end{bmatrix} \tag{3.92b}$$

by(3.29b), N=4 , k=1, $\alpha = 3$ and $\lambda = \frac{7}{2}$, we have

$$\overline{\Delta}^{\frac{7}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma((k+2)\alpha+1)}{\Gamma((k+2)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 216.649 & 0 & 0 \\ 0 & 0 & 0 & 1261 & 0 \\ 0 & 0 & 0 & 0 & 4015 \end{bmatrix} \tag{3.92c}$$

from substituting(3.92)(a),(b)and(c)in(3.91), we get

$$D^{\frac{7}{2}} \cong F^{(1)}.\overline{\Delta}^{\frac{7}{2}} \cdot G.\overline{T}_3(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2329 & 1255 & 48.529 & -24.265 & 9.17 \\ 27750 & 20920 & 7456 & 210.636 & -107.52 \\ 159400 & 136200 & 78780 & 23660 & 535.501 \end{bmatrix} . \begin{bmatrix} 1 \\ 2\,x^3-1 \\ 8\,x^6-8\,x^3+1 \\ 32\,x^9-48\,x^6+18\,x^3-1 \\ 128\,x^{12}-256\,x^9+160\,x^6-32\,x^3+1 \end{bmatrix} \tag{3.93}$$

$$D^{\frac{5}{2}} \cong F^{(1)}.\overline{\Delta}^{\frac{5}{2}} \cdot G.\overline{T}_2(x) \tag{3.94}$$

by(3.31), we have

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.383 & 0.721 & 0.0088 & -0.01 & 0.0047 \\ 1.052 & 0.708 & 0.182 & -0.0017 & -0.00077 \\ 0.88 & 0.663 & 0.268 & 0.044 & -0.00073 \end{bmatrix} \tag{3.95a}$$

from(3.29b), N=4 , k=1, $\alpha = 2$ and $\lambda = \frac{5}{2}$, we have

$$\bar{\Delta}^{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma((k+2)\alpha+1)}{\Gamma((k+2)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 18.054 & 0 & 0 \\ 0 & 0 & 0 & 61.9 & 0 \\ 0 & 0 & 0 & 0 & 140.056 \end{bmatrix} \tag{3.95b}$$

by substituting(3.92a),(3.95)(a)and(b)in(3.94), we get

$$D_{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 199.749 & 104.135 & 1.28 & -1.444 & 0.679 \\ 885.305 & 777.594 & 352.827 & 5.176 & -5.602 \\ 3.1\times10^{5} & 2.749\times10^{3} & 1.946\times10^{4} & 787.83 & 12.623 \end{bmatrix} . \begin{bmatrix} 1 \\ 2\,x^3-1 \\ 8\,x^6-8\,x^3+1 \\ 32\,x^9-48\,x^6+18\,x^3-1 \\ 128\,x^{12}-256\,x^9+160\,x^6-32\,x^3+1 \end{bmatrix} \tag{3.96}$$

from using the first root of $x^{\alpha 1} = \frac{1}{2}$ of the polynomial $T^{\alpha 1}_{m+1-\lambda}(x)$,

substituting this root in equation(3.89), we obtain

$$4304.4446\,c_2 + 14909.0146\,c_3 + 104608.456\,c_4 + (c_0 - c_2 + c_4)^3 = 6.1593 \tag{3.97}$$

by(3.90) , we have

$$u(0) = c_0 - c_1 + c_2 - c_3 + c_4 = 0 \tag{3.98a}$$

$$u^{(1)}(1) = 6\,c_1 + 24\,c_2 + 54\,c_3 + 96\,c_4 = 3 \tag{3.98b}$$

$$u^{(2)}(1) = 12\,c_1 + 192\,c_2 + 972\,c_3 + 3072\,c_4 = 6 \tag{3.98c}$$

$$u^{(3)}(1) = 12\,c_1 + 912\,c_2 + 10476\,c_3 + 58944\,c_4 = 6 \tag{3.98d}$$

from(3.97),(3.98)(a),(b)and(c), we get

$c_0 = 0.4947$, $c_1 = 0.4968$, $c_2 = 0.0016$, $c_3 = -0.0005$, $c_4 = 0.0001$.

Then, the approximate solution is

$$y(x) = 0.4947 + 0.4968(2\,x^3 - 1) + 0.0016(8\,x^6 - 8\,x^3 + 1) - 0.0005(32\,x^9 - 48\,x^6 + 18\,x^3 - 1) + 0.0001(128\,x^{12} - 256\,x^9 + 160\,x^6 - 32\,x^3 + 1$$

$$= 0.0128\,x^{12} - 0.416\,x^9 + 0.0528x^6 + 0.9686x^3 + 0.0001$$

Table(3.3)

x	Approximate solution	Exact solution
0.1	0.0010	0.001
0.2	0.0078	0.008

0.3	0.026	0.027
0.4	0.062	0.064
0.5	0.121	0.125
0.6	0.208	0.216
0.7	0.322	0.343
0.8	0.455	0.512
0.9	0.577	0.729

Example(3.3.17):

Consider the following multi-fractional order nonlinear differential equation,

$$D^{5}u(x)+D^{\frac{9}{2}}u(x)+D^{\frac{7}{2}}u(x)+u^{(4)}(x)=27.081\,x^{\frac{1}{2}}+x^{16} \qquad (3.99)$$

with mixed boundary condition,

$$u(0)=0\ ,u^{(1)}(1)=4\ ,\ u^{(2)}(1)=12\,,\ \ u^{(3)}(1)=24\,,\ u^{(4)}(0)=24 \qquad (3.100)$$

To solve the approximate solution with $m=5$, and $\alpha_1=4$, $\alpha_2=3$ the couple order for shifted Chebyshev polynomial of third kind such that the exact solution $(x)=x^4$.

By using equation(3.31) and lemma(3.3.3) and equation(3.43), we get

$$D^{\frac{9}{2}} \cong F^{(3)}.\bar{\Delta}^{\frac{9}{2}}\cdot H.\bar{V_4}(x) \qquad (3.101)$$

also, by(3.24), we get

$$\begin{bmatrix} 1 \\ 4\ x^4-3 \\ 16\ x^8-20\ x^4+5 \\ 64\ x^{12}-112\ x^8+56\ x^4-7 \\ 256\ x^{16}-576\ x^{12}+432\ x^8-120\ x^4+9 \\ 1024\ x^{20}-2816\ x^{16}+2816\ x^{12}-1232\ x^8+220\ x^4-11 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 & 0 \\ 9 & -120 & 432 & -576 & 256 & 0 \\ -11 & 220 & -1232 & 2816 & -2816 & 1024 \end{bmatrix}\begin{bmatrix} 1 \\ x^4 \\ x^8 \\ x^{12} \\ x^{16} \\ x^{20} \end{bmatrix}$$

$$F^{(3)}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 & 0 \\ 5 & -20 & 16 & 0 & 0 & 0 \\ -7 & 56 & -112 & 64 & 0 & 0 \\ 9 & -120 & 432 & -576 & 256 & 0 \\ -11 & 220 & -1232 & 2816 & -2816 & 1024 \end{bmatrix} \qquad (3.102a)$$

from(3.31), we have

$$H=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4.152 & -6.442 & -10.033 & -13.999 & -18 & -22 \\ -4.724 & -6.767 & -10.126 & -14.008 & -18 & -22 \\ -5.235 & -7.096 & -10.257 & -14.035 & -18.002 & -22 \\ -5.7 & -7.421 & -10.411 & -14.082 & -18.01 & -22 \end{bmatrix} \qquad (3.102b)$$

by(3.29b), N=5 ,k=1 , $\alpha_1=4$ and $\lambda=\frac{9}{2}$, we get

$$\bar{\Delta}^{\frac{9}{2}}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma((k+2)\alpha+1)}{\Gamma((k+2)\alpha+1-\lambda)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma((k+3)\alpha+1)}{\Gamma((k+3)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3466 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3413 & 0 & 0 \\ 0 & 0 & 0 & 0 & 152900 & 0 \\ 0 & 0 & 0 & 0 & 0 & 468800 \end{bmatrix} \qquad (3.102c)$$

from substituting(3.102)(a),(b)and(c)in(3.101), we obtain

$$D^{\frac{9}{2}} \cong F^{(3)}.\bar{\Delta}^{\frac{9}{2}} \cdot H.\overline{V_4}(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2.303\times 10^{5} & -3.572\times 10^{5} & -5.564\times 10^{5} & -7.763\times 10^{5} & -9.982\times 10^{5} & -1.22\times 10^{6} \\ 5.799\times 10^{5} & 1.023\times 10^{6} & 1.683\times 10^{6} & 2.375\times 10^{6} & 3.056\times 10^{6} & 3.735\times 10^{6} \\ -2.018\times 10^{8} & -2.741\times 10^{8} & -3.966\times 10^{8} & -5.428\times 10^{8} & -6.962\times 10^{8} & -8.508\times 10^{8} \\ -5.099\times 10^{8} & -5.447\times 10^{8} & -6.36\times 10^{8} & -7.919\times 10^{8} & -9.908\times 10^{8} & -1.206\times 10^{9} \end{bmatrix}$$

$$\cdot\begin{bmatrix} 1 \\ 4\ x^4-3 \\ 16\ x^8-20\ x^4+5 \\ 64\ x^{12}-112\ x^8+56\ x^4-7 \\ 256\ x^{16}-576\ x^{12}+432\ x^8-120\ x^4+9 \\ 1024\ x^{20}-2816\ x^{16}+2816\ x^{12}-1232\ x^8+220\ x^4-11 \end{bmatrix}$$

(3.103)

$$D^{\frac{7}{2}} \cong F^{(3)}.\bar{\Delta}^{\frac{7}{2}} \cdot H.\overline{V_3}(x) \qquad (3.104)$$

by(3.32), we have

$$H=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4.074 & -6.403 & -10.024 & -13.999 & -18 & -22 \\ -4.656 & -6.726 & -10.112 & -14.005 & -18 & -22 \\ -5.174 & -7.055 & -10.239 & -14.031 & -18.001 & -22 \\ -5.644 & -7.381 & -10.391 & -14.075 & -18.008 & -22 \end{bmatrix}$$

(3.105a)

from(3.29b), N=5 , k=1 , $\alpha_2 = 3$ and $\lambda = \frac{7}{2}$, we get

$$\bar{\Delta}^{\frac{7}{2}}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma((k+2)\alpha+1)}{\Gamma((k+2)\alpha+1-\lambda)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma((k+3)\alpha+1)}{\Gamma((k+3)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 216.649 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1261 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4015 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9556 \end{bmatrix} \quad (3.105b)$$

by substituting(3.102a),(3.105)(a)and(b)in(3.104), we obtain

$$D^{\frac{7}{2}} \cong F^{(3)}.\bar{\Delta}^{\frac{7}{2}} \cdot H.\bar{V}_3(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.412\times10^4 & -2.22\times10^4 & -3.475\times10^4 & -4.853\times10^4 & -6.239\times10^4 & -7.626\times10^4 \\ -2.769\times10^5 & -3.874\times10^5 & -5.728\times10^5 & -7.906\times10^5 & -1.016\times10^5 & -1.242\times10^5 \\ -2.318\times10^6 & -2.965\times10^6 & -4.118\times10^6 & -5.559\times10^6 & -7.113\times10^6 & -8.692\times10^6 \\ -1.218\times10^7 & -1.463\times10^7 & -1.915\times10^7 & -2.509\times10^7 & -3.18\times10^7 & -3.879\times10^7 \end{bmatrix}$$

$$. \begin{bmatrix} 1 \\ 4\ x^4 - 3 \\ 16\ x^8 - 20\ x^4 + 5 \\ 64\ x^{12} - 112\ x^8 + 56\ x^4 - 7 \\ 256\ x^{16} - 576\ x^{12} + 432\ x^8 - 120\ x^4 + 9 \\ 1024\ x^{20} - 2816\ x^{16} + 2816\ x^{12} - 1232\ x^8 + 220\ x^4 - 11 \end{bmatrix} \quad (3.106)$$

from using the first root of $x^{\alpha 1} = \frac{3}{4}$ of the polynomial $V_{m+1-\lambda}^{\alpha 1}(x)$,

substituting this root in equation(3.99), we obtain

$$-37936.2532\, c_2 + 3181500 c_3 - 84271000 c_4 - 160580000 c_5 + (c_0 - c_2 - c_3 + c_5)^4 = 26.4464 \quad (3.107)$$

by(3.100), we have

$$c_0 - 3\, c_1 + 5\, c_2 - 7\, c_3 + 9\, c_4 - 11\, c_5 = 0 \quad (3.108a)$$

$$16\, c_1 + 48\, c_2 + 96\, c_3 + 160\, c_4 + 240\, c_5 = 4 \quad (3.108b)$$

$$48\, c_1 + 656\, c_2 + 2848\, c_3 + 8160\, c_4 + 18640 c_5 = 12 \quad (3.108c)$$

$$96\, c_1 + 4896\, c_2 + 48192\, c_3 + 242112\, c_4 - 2505112\, c_5 = 24 \quad (3.108d)$$

$$96\, c_1 - 480\, c_2 + 1344\, c_3 - 2880\, c_4 + 5280\, c_5 = 24 \quad (3.108e)$$

from(3.107),(3.108)(a),(b),(c),(d)and(e), we obtain

$c_0 = 0.7500$, $c_1 = 0.2500$, $c_2 = c_3 = c_5 = 0$.

Then, the approximate solution is

$$y(x) = 0.7500 + 0.2500(4\, x^4 - 3) + 0$$
$$= 0.7500 + x^4 - 0.7500 = x^4$$

Example(3.3.18):

Consider the following multi-fractional order nonlinear differential equation,

$$D^3 u(x) + D^{\frac{5}{2}} u(x) + D^{\frac{3}{2}} u(x) + u^{(2)}(x) = 2.257\, x^{\frac{1}{2}} + x^4 \quad (3.109)$$

with mixed boundary condition,

$$u(0) = 0\ , u^{(1)}(1) = 2\ ,\ u^{(2)}(1) = 2 \quad (3.110)$$

To find the approximate solution with $m = 3$, and $\alpha_1 = 2$, $\alpha_2 = 1$ the couple order for shifted Chebyshev polynomial of fourth kind such that the exact solution is $(x) = x^2$.

By using equation(3.32) and lemma(3.3.3)and equation(3.44), we have

$$D^{\frac{5}{2}} \cong F^{(4)}.\bar{\Delta}^{\frac{5}{2}}.R.\overline{w_2}(x) \qquad (3.111)$$

also, from(3.25), we have

$$\overline{W_\alpha}(x) = F^{(4)}X_2$$

$$\begin{bmatrix} 1 \\ 2\,x^{\alpha 1}+1 \\ 8\,x^{2\alpha 1}-3 \\ 32\,x^{3\alpha 1}-16\,x^{2\alpha 1}-14\,x^{\alpha 1}+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -3 & 0 & 8 & 0 \\ 5 & -14 & -16 & 32 \end{bmatrix} \begin{bmatrix} 1 \\ x^2 \\ x^4 \\ x^6 \end{bmatrix}$$

$$F^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -3 & 0 & 8 & 0 \\ 5 & -14 & -16 & 32 \end{bmatrix} \qquad (3.112a)$$

by(3.32), we get

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.318 & -0.607 & -0.26 & -0.078 \\ -0.145 & -0.318 & -0.041 & 0.037 \end{bmatrix} \qquad (3.112b)$$

from(3.29b), N=3 , k=1, $\alpha_1 = 2$ and $\lambda = \frac{5}{2}$, we have

$$\bar{\Delta}^{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 18.054 & 0 \\ 0 & 0 & 0 & 61.9 \end{bmatrix} \qquad (3.112c)$$

by substituting(3.112)(a),(b)and(c)in(3.111), we get

$$D^{\frac{5}{2}} \cong F^{(4)}.\bar{\Delta}^{\frac{5}{2}}.R.\overline{w_2}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -45.923 & -200.39 & -396.606 & -615.233 \\ -194.426 & -973.324 & -2.085\times 10^3 & -3.406\times 10^3 \end{bmatrix} \begin{bmatrix} 1 \\ 2\,x^2+1 \\ 8\,x^4-3 \\ 32\,x^6-16\,x^4-14\,x^2+5 \end{bmatrix} \qquad (3.113)$$

$$D^{\frac{3}{2}} \cong F^{(4)}.\bar{\Delta}^{\frac{3}{2}}.R.\overline{w_1}(x) \qquad (3.114)$$

from(3.32), we get

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.424 & -0.764 & 0.497 & -0.263 \\ -0.17 & -0.364 & -0.0080 & 0.039 \end{bmatrix} \qquad (3.115a)$$

by(3.29b), N=3 , k=1, $\alpha_2 = 1$ and $\lambda = \frac{3}{2}$, we have

$$\bar{\Delta}^{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\lambda)} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\lambda)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2.257 & 0 \\ 0 & 0 & 0 & 4.514 \end{bmatrix}$$

(3.115b)

from substituting(3.112)(a),(3.115)(a)and(b) in(3.114), we get

$$D^{\frac{3}{2}} \cong F^{(4)}.\bar{\Delta}^{\frac{3}{2}}.R.\overline{w_1}(x) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -7.662 & -32.182 & -62.394 & -95.524 \\ -9.195 & -51.229 & -114.571 & -191.606 \end{bmatrix} \begin{bmatrix} 1 \\ 2\,x^2+1 \\ 8\,x^4-3 \\ 32\,x^6-16\,x^4-14\,x^2+5 \end{bmatrix}$$

(3.116)

by using the first root of $x^{\alpha 1} = \frac{1}{2}$ of the polynomial $W_{m+1-\lambda}^{\alpha 1}(x)$,

substituting this root in equation(3.110), we have

$$-2301.908c_2 - 11309.005\,c_3 + (c_0 - c_2 + 4\,c_3)^2 = 2.1477 \quad (3.117)$$

from(3.110), we get

$$u(0) = c_0 + c_1 - 3c_2 + 5\,c_3 = 0 \quad (3.118a)$$

$$u^{(1)}(1) = 4\,c_1 + 32\,c_2 + 100\,c_3 = 2 \quad (3.118b)$$

$$u^{(2)}(1) = 4\,c_1 + 96\,c_2 + 740\,c_3 = 2 \quad (3.118c)$$

by(3.117),(3.118)(a),(b)and(c), we obtain

$c_0 = -0.5145$, $c_1 = 0.5089$, $c_2 = -0.0016$, $c_3 = 0.0002$.

Then, the approximate solution is

$$y(x) = -0.5145 + 0.5089(2\,x^2+1) - 0.0016(8\,x^4-3) + 0.0002(32\,x^6 - 16\,x^4 - 14\,x^2 + 5)$$
$$= 0.0002 + 0.0064\,x^6 + 0.0096\,x^4 + 1.015\,x^2$$

Table (3.4)

x	Approximate solution	Exact solution
0.1	0.01	0.01
0.2	0.041	0.04
0.3	0.092	0.09
0.4	0.163	0.16
0.5	0.255	0.25
0.6	0.367	0.36
0.7	0.501	0.49
0.8	0.655	0.64
0.9	0.832	0.81

Conclusions

1. The operational matrices of fractional derivative for higher fractional order with different values are obtain in multi-order differential nonlinear equations which are given to support the best numerical solution and sometime exact solution for the considered example.

2. The fractional operational matrices of fractional derivative of some types of fractional chebyshev polynomial depending on value of the orders have been given to support the mixed boundary values multi-fractional nonlinear order to obtain the best numerical or exact solution some time.

3. The difficulty of driven the all types of matrices treated in details without any mistake in analytically and computationally in proving.

4. The relation between the types of operational matrices of wavelets chebyshev polynomials for fractional derivative has been presented for first time with new formulation of wavelets chebyshev types.

Future Work

1. Solving multi-order differential equations by using fractional lager polynomials may be taken.
2. Solving fractional diffusion equation by using fractional polynomials such as Legendre , Lager and Chebyshev ,….
3. Coupled fractional polynomials for solving fractional partial differential equations.
4. All above points (1),(2),(3) which are important to drive the operational matrices to simplify the processing of solving any class of nonlinear differential problems.

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